Class I.

I.

Consider $\begin{cases} u_x = f & 0 \leq x \leq 1. \\ u(0) = a \end{cases}$

$\Rightarrow u(x) = a + \int_0^x f(\xi)\, d\xi$

1. 差分法.

$\Rightarrow$ solve $u_0, u_1, \cdots, u_n$.

Let $\begin{cases} u_x\big|_{x=x_j} \approx \dfrac{u_j - u_{j-1}}{h} \\ \text{OR} \\ u_x\big|_{x=x_j} = \dfrac{u_{j+1} - u_j}{h} \end{cases}$ (Not right).

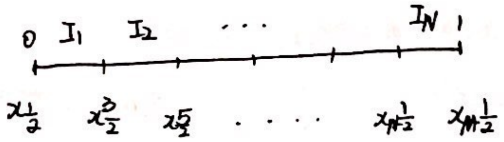
(upwind) $\Rightarrow \begin{cases} \dfrac{u_j - u_{j-1}}{h} = f(x_j) \\ u_0 = a \end{cases}$, i.e. $\begin{aligned} u_1 &= u_0 + h f(x_1) \\ u_2 &= u_1 + h f(x_2) \\ &\vdots \\ u_n &= u_{n-1} + h f(x_n) \end{aligned}$ "1st order"

OR.

$\Rightarrow \begin{cases} \dfrac{u_{j+1} - u_{j-1}}{2h} = f(x_j) \\ u_0 = a \end{cases}$ i.e. $\begin{aligned} u_1 - u_{-1} &= f(x_0) \cdot 2h \\ u_2 - u_0 &= f(x_1) \cdot 2h \\ &\vdots \end{aligned}$ "2nd order"

2. FEM-DG.

If $d=1$.



where $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$u_h(x) \in V_h$, $V_h = \{ v : v|_{I_j} \in P^k(I_j) \}$ (e.g. $k=1$.

)

do not need continuity.

$\dim(V_h) = (k+1)N$.

consider $\int_{I_j} u_x v\, dx = \int_{I_j} f v\, dx$

$\xRightarrow{\text{Intergration by parts.}}$ $-\int_{I_j} u v_x\, dx + uv\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} = -\int_{I_j} u v_x\, dx + u_{j+\frac{1}{2}} v_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} v_{j-\frac{1}{2}}$

$\Rightarrow -\int_{I_j} u v_x\, dx + u_{j+\frac{1}{2}} v_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} v_{j-\frac{1}{2}} = \int_{I_j} f v\, dx$

$\Rightarrow -\int_{I_j} u_h v_x\, dx + u_{h_{j+\frac{1}{2}}} v_{j+\frac{1}{2}} - u_{h_{j-\frac{1}{2}}} v_{j-\frac{1}{2}} = \int_{I_j} f v\, dx \qquad u_h \in V_h$

If $k=0$. Take $v = \begin{cases} 1 & x \in I_j \\ 0 & \text{otherwise!} \end{cases}$ we have $-\int_{I_j} u_h v_x\, dx = 0$ $(v_x|_{I_j} = 0)$.

Choose $u^-_{h_{j+\frac{1}{2}}} v^-_{j+\frac{1}{2}} - u^-_{h_{j-\frac{1}{2}}} v^+_{j-\frac{1}{2}} = \int_{I_j} f v\, dx \approx f(x_j) h$.

then we can get $\dfrac{u_j - u_{j-1}}{h} = f(x_j) \Rightarrow$ 差分格式.

Hence, the FEM-DG is as follows:

Find $u_h \in V_h$, s.t.

$$-\int_{I_j} u_h v_x dx + u_{h_{j+\frac{1}{2}}}^{-} v_{j+\frac{1}{2}}^{-} - u_{h_{j-\frac{1}{2}}}^{-} v_{j-\frac{1}{2}}^{+} = \int_{I_j} f v dx \qquad ⊛$$

H.W. 1.

(1) Code up the DG scheme (*) for $k=0,1,2,\ldots$ $h=\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \ldots$

where $f(x)=\cos x$, $a=0$ $(u(x)=\sin x)$.

Document: L^1 & L^∞ errored numerical order of accuracy.

$$e_h = Ch^r$$

$$e_{2h} = c\cdot(2h)^r$$

$$\Rightarrow r = \frac{\log\left(\frac{e_{2h}}{e_h}\right)}{\log 2}$$

	L^1	order	L^∞	order
$\frac{1}{10}$				
$\frac{1}{20}$				

basis of V_h:

local: $\varphi_j^l(x)$ $\begin{cases} \varphi_j^l(x)=0 \text{ if } x \notin I_j \\ l=0,1,\ldots,k,\ \varphi_j^k(x) \text{ is a basis of } P^k(I_j) \\ \text{i.e. } \left\{1, \frac{x-x_j}{h}, \frac{(x-x_j)^2}{h^2}, \cdots\right\}. \end{cases}$ ← 该basis 的条件数 不太好

Since $u_h(x) = \sum_{l=0}^{k} a_j^l \varphi_j^l(x)$ if $x \in I_j$.

For $j=1$, take $v=\varphi_1^m(x)$, then

$$-\int_{I_1} \sum_{l=0}^{k} a_1^l \varphi_1^l(x)(\varphi_1^m(x))_x dx + \sum_{l=0}^{k} a_1^l \varphi_1^l(x_{\frac{3}{2}}^-)\cdot\varphi_1^m(x_{\frac{3}{2}}^-)$$

$$\underbrace{-\sum_{l=0}^{k} a_1 \varphi_1^l(x_{\frac{1}{2}}^-)}_{u_h(x_{\frac{1}{2}}^-)=a} \varphi_1^m(x_{\frac{1}{2}}^+) = \int_{I_1} f \varphi_1^m(x) dx.$$

Let $u_j = \begin{bmatrix} a_j^0 \\ \vdots \\ a_j^k \end{bmatrix}$, then we have $\underset{\substack{\uparrow \\ (k+1)\times(k+1)}}{A} u_j = b$, where $\begin{cases} A_{ml} = -\int_{I_1} \varphi_1^l(x)(\varphi_1^m(x))_x dx + \varphi^l(x_{\frac{3}{2}}^-)\varphi_1^m(x_{\frac{3}{2}}^-) \\ b = a\varphi_1^m(x_{\frac{1}{2}}^+) + \int_{I_1} f\varphi_1^m(x) dx \end{cases}$

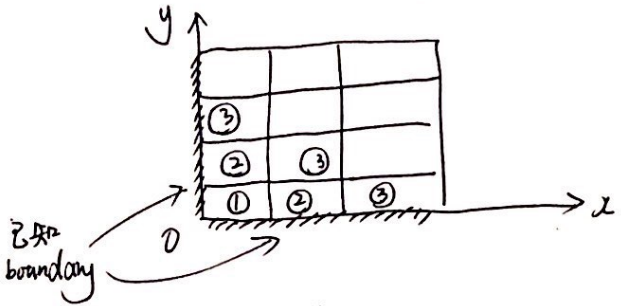
在DG中，一个区间一个区间解方程.

Remark: (a) A is invertible.

(b) $\|u_h\|_{L^2} \le C\|f\|$ ($a=0$, c is constant, 与f, h无关)

(c) $\|u-u_h\| \le Ch^{k+1}$ (c depends on $\|u\|_{H^{k+1}}$).

For $d=2$. consider $au_x + bu_y = f$, $a, b>0$, $(x,y)\in[0,1]^2$, $\begin{cases} u(x,0)=g(x) \\ u(0,y)=h(y). \end{cases}$



Consider Q^K polynomail space (各分量最多为k).

we have $\|u-u_h\| \le Ch^{k+\frac{1}{2}}$ (triangle)

Remark: (a) Richter: 若三角形取的好，则能证得 Ch^{k+1} 误差.

(b) Cockburn-Dong-Guzman: 若三角形都只有一个入流，则为 Ch^{k+1} 误差.

缺点:

1 Nonlinear problem:

$$(g(u))_x = f. \quad \text{e.g.} \quad \left(\frac{u^2}{2}\right)_x = f \Rightarrow \underline{u}\cdot u_x = f$$

u正负不定，"风向不定".

2. 方程组

For $Au_x = f$, A is a matrix. e.g.1. $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}f_1\\f_2\end{pmatrix}$.

$\therefore (u+v)_x = f_1+f_2, \quad -(u-v)_x = -f_1+f_2$.

e.g.2. $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}_x + \begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}f_1\\f_2\end{pmatrix}$

H.W.2. $(d=1)$

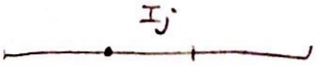
(*): (2) ① Prove the DG method ⊗ is well-defined, i.e. A is invertible.

② Prove stability $\|u_h\| \le C\|f\|$ for $a=0$.

proof: part ①:

Let $u(x) = \sum_{l=0}^{k} a_j^l \varphi_j^l(x)$. $x\in I_j$.

then $\begin{cases} A_{ml} = \int_{I_j} \varphi_j^l \varphi_j^m dx + \varphi_j^l(x_{j+\frac{1}{2}}^-)\varphi_j^m(x_{j+\frac{1}{2}}^-) \\ b_m = \underset{\substack{\uparrow \\ \text{计 } j=1.\ a_0=a \text{ (左端点)}}}{a_{j-1}} \varphi_j^m(x_{j-\frac{1}{2}}^+) + \int_{I_j} f \varphi_j^m(x) dx \end{cases}$



Ⅱ. Consider $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}$ with periodic boundary condition.

1. Definition

After semi-discrete, we can choose <u>Runge-kutta in time</u> (Runge-kutta-DG (RKDG)).

① Discrete scheme.

$$y_{n+1} = y_n + h\phi.$$

First, consider DG in semi-discrete:

choose $V_h = \{ v : v|_{I_j} \in P^k(I_j) \}$.

DG. $\begin{cases} \text{Find } u_h(\cdot, t) \in V_h \text{ s.t.} \\ \boxed{\int_{I_j} (u_h)_t v \, dx - \int_{I_j} f(u_h) v_x \, dx + \underbrace{\hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}} = 0} \\ \text{where } \underset{\substack{\uparrow \\ \text{numerical flux}}}{\hat{f}_{j+\frac{1}{2}}} = \hat{f}(u^-_{hj+\frac{1}{2}}, u^+_{hj-\frac{1}{2}}). = u^-_{j+\frac{1}{2}} \dashrightarrow \underline{u^-_{hj+\frac{1}{2}} v^-_{j+\frac{1}{2}} - u^-_{hj-\frac{1}{2}} v^+_{j-\frac{1}{2}}} \end{cases}$

scheme Ⅰ.

e.g. If we choose $\phi = f$ in Runge-kutta DG, we have Euler forward scheme:

$$\int_{I_j} \frac{u_h^{n+1} - u_h^n}{\Delta t} \cdot v \, dx - \int_{I_j} f(u_h^n) v_x \, dx + \hat{f}^n_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0.$$

Now consider $k = 0$, rewrite $\overset{u_h(k,t)}{=} u_h(0,t) = u_j(t)$, $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$.

Take $v = \begin{cases} 1 & x \in I_j \\ 0 & \text{otherwise}. \end{cases}$

From (DG), we have $h u_j'(t) + \hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} = 0$

$$\Rightarrow \quad u_j'(t) + \frac{1}{h}\left[\hat{f}(u_j(t), u_{j+1}(t)) - \hat{f}(u_{j-1}(t), u_j(t))\right] = 0 \qquad \circledast\circledast$$

where $\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})$ should satisfies

(a) $\hat{f}(u,u) = f(u)$

(b) $\hat{f}$ is Lipschitz continuous (L_1, L_2)

c) $\hat{f}(\uparrow, \downarrow)$.

with condition (a) - (c), $\circledast\circledast$ is monotonous flux scheme.

Class 2.

Recall $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}$

②

- Existence of discountinous in the solution (Even if $u^0(0)$ is smooth)

$u' \notin C^0$

$\Rightarrow$ strong solution 不存在.

- Weak Solution : 存在, 但无唯一性.

- Entropy solution : $\exists !$

满足 $U(u)_t + F(U)_x \leq 0$ 不等式的 weak solution is entropy solution.

Entropy condition : $U''(u) \geq 0$.

e.g. Let $\begin{cases} U(u) = u^2/2 \\ F'(u) = U'(u) f'(u) \end{cases}$ then $U(u)_t + F(U)_x = U'(u)u_t + U'(u)f'(u)u_x$

$= U'(u)[\overset{=0}{u_t + \underset{\overset{\|}{f(u)_x}}{f'(u)u_x}}] = 0$

③ <u>cell entropy inequality for DG.</u>

Chose $U(u) = \frac{u^2}{2}$ and take $v = u_h \in V_h$.

then by (DG) <u>schemeI</u>

$$\int_{I_j} (u_h)_t u_h \, dx - \int_{I_j} f(u_h)(u_h)_x dx + \hat{f}_{j+\frac{1}{2}} (u_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} (u_h)^+_{j-\frac{1}{2}} = 0$$

$$\Rightarrow \frac{d}{dt}\int_{I_j} U(u_h) \, dx - \int_{I_j} f(u_h)(u_h)_x dx + \hat{f}_{j+\frac{1}{2}} (u_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} (u_h)^+_{j-\frac{1}{2}} = 0$$

Define $g(u) = \int_{I_j} f(v) dv$, then $g'(u) = f(u)$

since $f(u_h)(u_h)_x = (g(u_h))_x$.

then $-\int_{I_j} f(u_h)(u_h)_x dx = \int_{I_j} (g(u_h))_x dx = -g((u_h)^-_{j+\frac{1}{2}}) + g((u_h)^+_{j-\frac{1}{2}})$

$$\Rightarrow \overbrace{\frac{d}{dt}\int_{I_j} U(u_h) dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}}}^{(1)} + \overbrace{\hat{F}_{j-\frac{1}{2}} + g((u_h)^+_{j-\frac{1}{2}}) - \hat{f}_{j-\frac{1}{2}} (u_h)^+_{j-\frac{1}{2}}}^{\Theta_{j-\frac{1}{2}}} = 0$$

where $\hat{F}_{j+\frac{1}{2}} = -g((u_h)^-_{j+\frac{1}{2}}) + \hat{f}_{j+\frac{1}{2}} (u_h)^-_{j+\frac{1}{2}}$

we want to prove $\Theta_{j-\frac{1}{2}} \geq 0$, 即证 $(1) \leq 0$ $\Rightarrow$ satisfies entropy condition.

Now we need to prove that $\theta_{j-\frac{1}{2}} \geqslant 0$.

$$\theta_{j-\frac{1}{2}} = -g((u_h)^-_{j-\frac{1}{2}}) + \hat{f}_{j-\frac{1}{2}}(u_h)^-_{j-\frac{1}{2}} + g((u_h)^+_{j-\frac{1}{2}}) - \hat{f}_{j-\frac{1}{2}}((u_h)^+_{j-\frac{1}{2}})$$

$$=: -g(u^-) + \hat{f}\cdot u^- + g(u^+) - \hat{f}\cdot u^+$$

$$= g(u^+) - g(u^-) - \hat{f}(u^+ - u^-)$$

$$= g'(\xi)(u^+ - u^-) - \hat{f}(u^+ - u^-)$$

$\xi \in (u^-, u^+)$, $g' = f$

$$= (f(\xi) - \hat{f})(u^+ - u^-)$$

$$\overset{(a)\text{ of } \hat{f}}{=} (\hat{f}(\xi,\xi) - \hat{f})(u^+ - u^-)$$

$$= [\underline{\hat{f}(\xi,\xi) - \hat{f}(u^-,\xi)} + \underset{?}{\underline{\hat{f}(u^-,\xi) - \hat{f}(u^-,u^+)}}](u^+ - u^-) \qquad \circledast.$$

(i) If $u^- < u^+$, then $u^- \leqslant \xi \leqslant u^+$

then $\begin{cases} \hat{f}(\xi,\xi) - \hat{f}(u^-,\xi) \geqslant 0 \\ \hat{f}(u^-,\xi) - \hat{f}(u^-,u^+) \geqslant 0 \end{cases}$ (by (c) of $\hat{f}$). $\Rightarrow$ $\circledast$式 $\geqslant 0$.

(ii) If $u^- > u^+$, then $u^- \geqslant \xi \geqslant u^+$.

then $u^+ - u^- \leqslant 0$

then $\begin{cases} \hat{f}(\xi,\xi) - f(u^-,\xi) \leqslant 0 \\ \hat{f}(u^-,\xi) - \hat{f}(u^-,u^+) \leqslant 0 \end{cases}$ (by (c) of $\hat{f}$) $\Rightarrow$ $\circledast$式 $\geqslant 0$.

Hence. $\theta_{j-\frac{1}{2}} \geqslant 0$.

Hence $\frac{d}{dt}\int_{I_j} U(u_h)\,dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \leqslant 0$. (2)

□

2. Stability, Optimal error estimate.

① Stabilitity.

By inequality (2). and take $\sum_j (2)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_h)^2\,dx \underbrace{- \hat{F}_{\frac{1}{2}} + \hat{F}_{N+\frac{1}{2}}}_{\text{periodic boundary "=0"}} \leqslant 0 \qquad (3)$$

(3)关于t积分得 $\int_0^1 u_h^2(x,t)\,dt \leqslant \int_0^1 u_h^2(x,0)\,dx$. $\Rightarrow$ stability.

② optimal error estimate.

$$\| u - u_h \|_{L^2} \le C h^{k+1}.$$

proof: part I. Two facts:

<u>Fact I</u>: u satisfies scheme I.

since $u_t + f(u)_x = 0$,

then $\int_{I_j} u_t v \, dx - \int_{I_j} f(u) v_x + \underline{\hat{f}(u^-_{j+\frac12}, u^+_{j+\frac12})} v^-_{j+\frac12} - \underline{\hat{f}(u^-_{j-\frac12}, u^+_{j-\frac12})} v^+_{j-\frac12} = 0$ holds.

$$\Big\downarrow u^-_{j+\frac12} = u^+_{j+\frac12} \Leftarrow \text{since } u \text{ is smooth enough} \Rightarrow u^-_{j-\frac12} = u^+_{j-\frac12} \Big\downarrow$$

$$\hat{f}(u^-_{j+\frac12}, u^-_{j+\frac12}) = f(u^-_{j+\frac12}) \qquad \hat{f}(u^+_{j-\frac12}, u^+_{j-\frac12}) = f(u^+_{j-\frac12})$$

Hence, u satisfies

$$\int_{I_j} u_t v \, dx - \int_{I_j} f(u) v_x + f(u^-_{j+\frac12}) v^-_{j+\frac12} - f(u^+_{j-\frac12}) v^+_{j-\frac12} = 0, \ \forall v \in V_h. \quad (1)$$

<u>Fact 2</u>: Let $e = u - u_h$, e also satisfies scheme I.

If $f(u) = u$, $\hat{f}(u^-, u^+) = u^-$,

then

"Error equation" $$\int_{I_j} e_t v \, dx - \int_{I_j} e v_x \, dx + e^-_{j+\frac12} v^-_{j+\frac12} - e^-_{j-\frac12} v^+_{j-\frac12} = 0 \quad (2)$$

part II. error estimate for e.

Let $e = u - u_h = (u - Pu) - (u_h - Pu) =: \eta - \xi$

It should be noted that $\xi \in V_h$ (since $u_h \in V_h$, $Pu \in V_h$).

Take $v = \xi$ in error equation (2), we have

$$\text{(LHS)} \quad \int_{I_j} \xi_t \cdot \xi \, dx - \int_{I_j} \xi \cdot \xi_x \, dx + \xi^-_{j+\frac12} \xi^-_{j+\frac12} - \xi^+_{j-\frac12} \xi^+_{j-\frac12} \quad (3)$$

$$= \int_{I_j} \eta_t \cdot \xi \, dx - \int_{I_j} \eta \cdot \xi_x \, dx + \eta^-_{j+\frac12} \xi^-_{j+\frac12} - \eta^+_{j-\frac12} \xi^+_{j-\frac12}. \quad \text{(RHS)}.$$

For LHS of (3), we have

$$\text{LHS} = \frac12 \frac{d}{dt} \int_{I_j} \xi^2 dx + \hat{F}_{j+\frac12} - \hat{F}_{j-\frac12} + \Theta_{j-\frac12}.$$

For the RHS of (3).

Suppose $\begin{cases} \eta^-_{j+\frac{1}{2}} = 0 \quad (\text{i.e. } (u-Pu)^-_{j+\frac{1}{2}} = 0) \\ \int_{I_j} (u-Pu)\, v\, dx = 0, \quad \forall\, v \in P^{k-1}(I_j). \end{cases}$? then $\|u - Pu\| \le Ch^{k+1}$.

Then $RHS = \int_{I_j} \eta_t \xi \, dx$

Combining LHS & RHS, we have

$$\frac{1}{2}\frac{d}{dt}\int_{I_j} \xi^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}} = \int_{I_j} \eta_t \xi \, dx$$

$\Longrightarrow \sum_j$ to get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \xi^2 dx + \theta = \int_0^1 \eta_t \xi \, dx \le \|\xi\| \cdot \|\eta_t\|.$$

Since $\begin{cases} \frac{1}{2}\frac{d}{dt}\|\xi\|^2 = \|\xi\| \cdot \frac{d}{dt}\|\xi\| \\ \theta \geq 0 \end{cases}$

$$\Longrightarrow \quad \frac{d}{dt}\|\xi\| \le \underline{\|\eta_t\| \le Ch^{k+1}}$$

$(\ \|u_t - Pu_t\| \le Ch^{k+1}\ )$. ?

$$\Longrightarrow \quad \|\xi(\cdot,t)\| \le \underset{\overset{\|}{0}}{\|\xi(\cdot,0)\|} + Ct\,h^{k+1} = Cth^{k+1}$$

$$\Longrightarrow \quad \|e(\cdot,t)\| \le (1+Ct)\,h^{k+1}.$$

3. Implementation.

$$\varphi_j^l(x) \Rightarrow u_h(x,t) = \sum_{l=0}^{k} a_j^l(t)\,\varphi_j^l(x).$$

① Consider Euler forward:

$$\sum_{l=0}^{k} (a_j^l)^{n+1} \int_{I_j} \varphi_j^l(x)\varphi_j^m(x)\,dx = \sum_{l=0}^{k} (a_j^l)^n \int_{I_j} \varphi_j^l(x)\varphi_j^m(x)\,dx + \Delta t \int_{I_j} f\Big(\sum_{l=0}^{k} (a_j^l)^n \cdot \varphi_j^l(x)\Big) \cdot \varphi_j^m(x)\,dx$$
$$- \hat{f}^n_{j+\frac{1}{2}}\,\varphi_j^m(x^-_{j+\frac{1}{2}}) + \hat{f}^n_{j-\frac{1}{2}}\,\varphi_j^m(x^+_{j-\frac{1}{2}}).$$

where $\hat{f}^n_{j+\frac{1}{2}} = \hat{f}\Big(\sum_{l=0}^{k} (a_j^l)^n \varphi_j^l(x^-_{j+\frac{1}{2}}),\ \sum_{l=0}^{k} (a_{j+1}^l)^n \cdot \varphi_{j+1}^l(x^+_{j+\frac{1}{2}})\Big)$.

Let $u_j = \begin{bmatrix} a_j^0 \\ \vdots \\ a_j^k \end{bmatrix}$, then $M u_j^{n+1} = M u_j^n + \Delta t(\cdots\ )$

$\Rightarrow \quad u_j^{n+1} = u_j^n + \Delta t M^{-1}(\cdots)$

where M is mass metrix with $M_{ml} = \int_{I_j} \varphi_j^l(x)\varphi_j^m(x)\,dx$.

② Comparison

1st order Rung-Kutta: Euler forward.

$$u_t = L(u). \quad u^{n+1} = u^n + \Delta t L(u^n)$$

2nd order Rung-Kutta:

$$u^{(1)} = u^n + \Delta t L(u^n)$$

$$u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}\left(u^{(1)} + \Delta t L(u^{(1)})\right)$$

[此时可以使用 P^1 : $\max\limits_u |f'(u)| \frac{\Delta t}{\Delta x} \leq \frac{1}{3}$]

3nd order Rung-Kutta

$$u^{(1)} = u^n + \Delta t L(u^n)$$

$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}(u^{(1)} + \Delta t L(u^{(1)}))$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}(u^{(2)} + \Delta t L(u^{(2)})).$$

此时 [P^k: $\max\limits_u |f'(u)| \frac{\Delta t}{\Delta x} \leq \frac{1}{2k+1}$]

Class 3.

Recall that

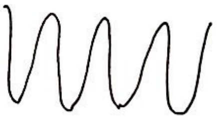
$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}$$

1. , Definition
2. , Stability, optimal error estimate
3. implementation
4. super convergence issue.

① in weaker norm. (The advantage of Galerkin Method).

negative norm : $\|u\|_{-k} := \sup\limits_{w} \dfrac{\langle u, w \rangle}{\|w\|_{H^k}}$

i.g. $u = \sin kx$



$\xRightarrow[\text{强震荡}]{}$ $\|\sin kx\|$ 较大.

since $\int_0^{2\pi} \sin kx \underset{\overset{\uparrow}{\text{smooth}}}{w(x)}\, dx \xrightarrow{k\to\infty} 0 \Rightarrow$ weaker norm 很小.

Hence, we wish $\|u - u_h\|_{-k} \le Ch^m$, where $m > k+1$, eg $m = 2k+1$. (cf. Cockbunn-Luchin-shu-Suli).

Example I:

Pu : L^2-projection.

$\int_{I_j} (u - Pu) v \, dx = 0 \quad \forall\, v \in P(I_j)$, Recall that $\begin{cases} \overset{\swarrow\ \text{error estimate in strong norm.}}{\|u - Pu\| \le Ch^{k+1}} \\ \|w - Pw\| \le Ch^{k+1} \end{cases}$

$$\left|\int_0^1 (u - Pu) w \, dx\right| = \left|\int_0^1 (u - Pu)(w - Pw)\right| \le \|u - Pu\| \, \|w - Pw\|$$

$$\le \underset{\overset{\nwarrow}{\text{error estimate in weak norm.}}}{Ch^{2k+2}}.$$

Example 2:

If with Uniform mesh :

Let $w_h = Qu_h$. , then $\|\underset{\overset{\uparrow}{u\text{ is smooth enough.}}}{u} - w_h\| \le Ch^{2k+1}$

(cf. J. Ryan)

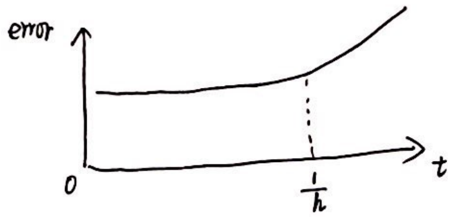
② In strong norm.

Pu is a projection of u in V_h

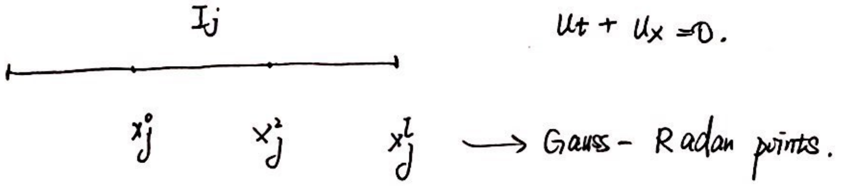
Q: $\|u_h - Pu\| \overset{?}{\leq} Ch^{k+2}$?

$$\|u-u_h\| \leq \|u-Pu\| + \|Pu-u_h\|$$
$$\leq \underset{\substack{\uparrow \\ \text{(not relevant to } t)}}{C_1 h^{k+1}} + C_2(1+t)h^{k+2} \left\{ \begin{array}{ll} \approx Ch^{k+1} & \text{If } t \leq \frac{1}{h} \\ \text{from } Ch^{k+1} \nearrow & \text{If } t >> \frac{1}{h} \end{array} \right.$$

(cf. Cheng, Y. Yang, ...).



③ At special points.



$u_t + u_x = 0.$

$$\|(u-u_h)(x_j^l)\| \leq Ch^{k+2}.$$
$$\|(u-u_h)(x_{j+\frac{1}{2}}^-)\| \leq Ch^{2k+1}.$$

(cf. Cao, ZM. Zhang, Y. Yang ...).

5. Limiter.

① Idea: 目前为止,DG不适用于激波解 (或非线性方程组) $\left\{ \begin{array}{l} \text{method 1: + 人工粘性项 (构造困难, 需要经验).} \\ \text{approach 2: + 限制器. (Limiter).} \end{array} \right.$

Solve $\int_{I_j} \hat{u}_h^{n+1} v\, dx = \int_{I_j} u_h^n v\, dx + \left[\int_{I_j} f(u_h^n) v_x dx - \hat{f}_{j+\frac{1}{2}}^n v_{j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}}^n v_{j+\frac{1}{2}}^+\right]\Delta t,\ v \in V_h.$

then Let. $\underset{\nwarrow}{u_h^{n+1}} = \text{limited}(\underset{\uparrow}{\hat{u}_h^{n+1}})$.

piecewise polynomial of degree of k.

(i) cell-average does not change:

$$\bar{u}_h^{n+1} = \bar{\hat{u}}_h^{n+1}. \qquad \text{where} \quad \bar{v}_j = \frac{1}{h}\int_{I_j} v(x)\,dx.$$

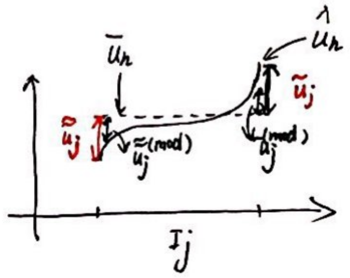
(ii).

② Example I. Minmod Limiter (generalized MUSCL Limiter).

1° Definition Define

$$\tilde{u}_j = u^{-}_{j+\frac{1}{2}} - \bar{u}_j$$

$$\tilde{\tilde{u}}_j = \bar{u}_j - u^{+}_{j-\frac{1}{2}}$$



Let $\tilde{u}_j^{(mod)} = m(\tilde{u}_j,\ \bar{u}_{j+1} - \bar{u}_j,\ \bar{u}_j - \bar{u}_{j-1})$.

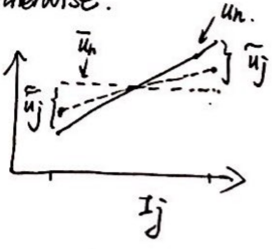
$\tilde{\tilde{u}}_j^{(mod)} = m(\tilde{\tilde{u}}_j,\ \bar{u}_{j+1} - \bar{u}_j,\ \bar{u}_j - \bar{u}_{j-1})$.

where $m(a_1, \cdots, a_m) = \begin{cases} s \min\limits_{i} |a_i|, & \text{if } \operatorname{sign}(a_1) = \operatorname{sign}(a_2) \cdots = \operatorname{sign}(a_m) = s. \\ 0, & \text{otherwise}. \end{cases}$

If $k=1$.

$$\tilde{u}_j = \tilde{\tilde{u}}_j,$$

$$\tilde{u}_j^{(mod)} = \tilde{\tilde{u}}_j^{(mod)}$$



$$\text{limited}(u) = \bar{u}_j + \tilde{u}_j^{(mod)} \frac{(x-x_j)}{2}.$$

If $k=2$ left point + right point + average is enough. for Freedom.

If $k \geq 3$

2° Theorem: The limited RKDG scheme is TVDM (total variation diminishing in the means).

$$TV(\bar{u}^{n+1}) \leq TV(\bar{u}^n).$$

where $$\underset{\text{Semi-norm.}}{\underset{\uparrow}{TV(\bar{u})}} = \sum_j (\bar{u}_{j+1} - \bar{u}_j)$$

proof: part 1. <u>Lemma</u> (Harten).

If $\bar{u}_j^{n+1} = \bar{u}_j^n + C_{j+\frac{1}{2}}(\bar{u}_{j+1}^n - \bar{u}_j^n) - D_{j-\frac{1}{2}}(\bar{u}_j^n - \bar{u}_{j-1}^n)$ (1)

and $C_{j+\frac{1}{2}} \geqslant 0$, $D_{j-\frac{1}{2}} \geqslant 0$, $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leqslant 1$

then the scheme is TVD.

proof: Replace j by j+1 in (1), we have

$$\bar{u}_{j+1}^{n+1} = \bar{u}_{j+1}^n + C_{j+\frac{3}{2}}(\bar{u}_{j+2}^n - \bar{u}_{j+1}^n) - D_{j+\frac{1}{2}}(\bar{u}_{j+1}^n - \bar{u}_j^n) \quad (2)$$

(2) - (1) to get

$$\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1} = \bar{u}_{j+1}^n - \bar{u}_j^n + C_{j+\frac{3}{2}}(\bar{u}_{j+2}^n - \bar{u}_{j+1}^n) \quad C_{j+\frac{1}{2}}(\bar{u}_{j+1}^n - \bar{u}_j^n)$$
$$- D_{j+\frac{1}{2}}(\bar{u}_{j+1}^n - \bar{u}_j^n) + D_{j\ \frac{1}{2}}(\bar{u}_j^n - \bar{u}_{j-1}^n)$$

$$= (1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}})(\bar{u}_{j+1}^n - \bar{u}_j^n) + C_{j+\frac{3}{2}}(\bar{u}_{j+2}^n - \bar{u}_{j+1}^n)$$
$$+ D_{j-\frac{1}{2}}(\bar{u}_j^n - \bar{u}_{j-1}^n)$$

then
$$\sum_j |\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1}| \leqslant \sum_j \underbrace{(1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}})}_{\geqslant 0} |\bar{u}_{j+1}^n - \bar{u}_j^n| + \sum_j \underbrace{\overset{\geqslant 0}{C_{j+\frac{3}{2}}}}_{\substack{\Uparrow \\ C_{j+\frac{1}{2}}}} \underbrace{|\bar{u}_{j+2}^n - \bar{u}_{j+1}^n|}_{\substack{\Uparrow \\ |\bar{u}_{j+1}^n - \bar{u}_{j+1}^n|}}$$
$$+ \sum_j \underbrace{\overset{\geqslant 0}{D_{j-\frac{1}{2}}}}_{\substack{\Uparrow \\ D_{j+\frac{1}{2}}}} \underbrace{|\bar{u}_j^n - \bar{u}_{j-1}^n|}_{\substack{\Uparrow \\ |\bar{u}_{j+1}^n - \bar{u}_j^n|}} \text{ (}\textstyle\sum_j\text{后不变).}$$

$$\Rightarrow \sum_j |\bar{u}_{j+1}^n - \bar{u}_j^n|.$$

part 2.

Recall that.

$$\int_{I_j} \hat{u}_h^{n+1} v\,dx = \int_{I_j} u_h^n v\,dx + \left[\int_{I_j} f(u_h^n) v_x\,dx - \hat{f}_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+\right]\Delta t, \ \forall v \in V_h. \quad (3)$$

Take $v=1$ in (3)

then
$$h\bar{u}_j^{n+1} = h\bar{u}_j^n - (\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}})\Delta t$$
$$\Rightarrow \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}})$$

Let $\frac{\Delta t}{\Delta x} = \lambda \longrightarrow$ $= \bar{u}_j^n - \lambda(\hat{f}(\underline{u_{j+\frac{1}{2}}^-}, \underline{u_{j+\frac{1}{2}}^+})) - \hat{f}(\underline{u_{j-\frac{1}{2}}^-}, \underline{u_{j-\frac{1}{2}}^+}))$

↖ limited 后的 ↗

$$=\bar{u}_j^n-\underbrace{\lambda\Big(\hat{f}(u^-_{j+\frac12},u^+_{j+\frac12})-\hat{f}(u^-_{j+\frac12},u^+_{j-\frac12})}_{C_{j+\frac12}(\bar{u}^n_{j+1}-\bar{u}^n_j)}+\underbrace{\hat{f}(u^-_{j+\frac12},u^+_{j-\frac12})-\hat{f}(u^-_{j-\frac12},u^+_{j-\frac12})}_{-D_{j-\frac12}(\bar{u}^n_j-\bar{u}^n_{j-1}).}\Big).$$

For $C_{j+\frac12}=-\lambda\dfrac{\hat{f}(u^-_{j+\frac12},u^+_{j+\frac12})-\hat{f}(u^-_{j+\frac12},u^+_{j-\frac12})}{\bar{u}^n_{j+1}-\bar{u}^n_j}$

suppose $\hat{f}$ is differentable.

$$=\underbrace{-\lambda.\ \hat{f}_2(u^-_{j+\frac12},\xi)}_{\geqslant 0}\frac{(u^+_{j+\frac12}-u^+_{j-\frac12})}{\bar{u}_{j+1}-\bar{u}_j}\quad (n\text{-step}).$$

since $\hat{f}(\uparrow,\downarrow)$.

$$=-\lambda\hat{f}_2(u^-_{j+\frac12},\xi)\ \frac{\bar{u}_{j+1}-\tilde{\tilde{u}}^{(mod)}_{j+1}-\bar{u}_j+\tilde{\tilde{u}}^{(mod)}_j}{\bar{u}_{j+1}-\bar{u}_j}$$

$$=\underbrace{-\lambda\hat{f}(u^-_{j+\frac12},\xi)}_{\geqslant 0}\Bigg(1-\underbrace{\underbrace{\frac{\tilde{\tilde{u}}^{(mod)}_{j+1}}{\bar{u}_{j+1}-\bar{u}_j}}_{\in[0,1]}+\underbrace{\frac{\tilde{\tilde{u}}^{(mod)}_{j}}{\bar{u}_{j+1}-\bar{u}_j}}_{\in[0,1]}}_{\in[0,2]}\Bigg)$$

$$\begin{cases}\geqslant 0\\ \\ \leq 2\lambda L_2\end{cases}\nwarrow \text{Lipschitz constant}.$$

Similarly, we have.

$D_{j-\frac12}\in[0,\ 2\lambda L_1]$

we can choose $\lambda\leq\dfrac{1}{2(L_1+L_2)}$ to satisfy $C_{j+\frac12}+D_{j+\frac12}\leq 1$. □

If u is smooth, $\bar{u}_j=u_j+O(h^2)$

$$\bar{u}_j=\frac{1}{h}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u(x)\,dx=\frac{1}{h}\int_{x_{j-\frac12}}^{x_{j+\frac12}}(u(x_i)+u_x(x_j)(x-x_j)+O(h^2))\,dx$$

$$=u(x_i)+O(h^2).$$

then $\tilde{u}_j=\bar{u}_{j+\frac12}-\bar{u}_j$

$$=u_{j+\frac12}-u_j+O(h^2)$$

$$=(u_j+u_x(x_j)\frac{h}{2}+O(h^2))+O(h^2)-u_j=u_x(x_j)\frac{h}{2}+O(h^2)$$

$$\tilde{u}_j=u_x(x_j)\frac{h}{2}.+O(h^2)$$

$$\bar{u}_{j+1} - \bar{u}_j = u_{j+1} - u_j + O(h^2) = u_x(x_j)h + O(h^2)$$

$$\bar{u}_j - \bar{u}_{j-1} = u_x(x_j)h + O(h^2)$$

$$\Rightarrow \quad m(\tilde{u}_j\ ,\ \bar{u}_{j+1} - \bar{u}_j\ ,\ \bar{u}_j - \bar{u}_{j-1})$$

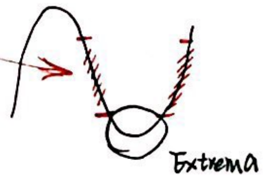
$$= m\left(u_x(x_j)\tfrac{h}{2} + O(h^2),\ u_x(x_j)h + O(h^2),\ u_x(x_j)h + O(h^2) \right)$$

3° Problem & Approvement.

If $u_x(x_j) = O(1)$

(assume u is smooth and monotone)

$$= \tilde{u}_j .$$



Similarly, $m(\tilde{\tilde{u}}_j,\ \bar{u}_{j+1} - \bar{u}_j,\ \bar{u}_j - \bar{u}_{j-1}) = \tilde{\tilde{u}}_j .$

Theorem (Osher) TVD schemes at most first order at smooth extrema.

If $e_{\overset{\uparrow}{3}} = h$, then $\frac{1}{N}\sum_j |e_j| \geq \frac{1}{N} h = h^2$ (L^1-error).
(extrema point)

L^1 - error of a TVD scheme can be at most 2^{nd} order. (Disadvantage of TVDM).

Approach to overcome the disadvantage of TVD.

TVBM scheme (Total variation bounded scheme).

$$TV(\bar{u}^{n+1}) \begin{cases} \leq TV(\bar{u}^n) + O(\Delta t) \\ \leq (1 + C\Delta t)\, TV(\bar{u}^n) \end{cases}$$

$$TV(\bar{u}^n) \leq \text{constant} \quad \text{for} \quad n\Delta t \leq T$$

How to implement?

M需要根据problem不同调整

Define $\bar{m}(a_1, \ldots, a_m) = \begin{cases} a_1, & \text{if } |a_1| \leq \underline{M}h^2 \text{ where } M: \text{constant}, \ \underline{M \approx \max|u^{0}_{xx}|} \\ & \text{(cf. shu math comp 87/ Cockburn-shu. 89).} \\ m(a_1, \ldots, a_m), & \text{otherwise.} \end{cases}$

then $\tilde{u}_j^{(mod)}$ and $\tilde{\tilde{u}}_j^{(mod)}$ can be replaced by

$$\tilde{u}_j^{(mod)} = \bar{m}(\tilde{u}_j,\ \bar{u}_{j+1} - \bar{u}_j,\ \bar{u}_j - \bar{u}_{j-1})$$

$$\tilde{\tilde{u}}_j^{(mod)} = \bar{m}(\tilde{\tilde{u}}_j,\ \bar{u}_{j+1} - \bar{u}_j,\ \bar{u}_j - \bar{u}_{j-1}).$$

and. $\tilde{u}_j^{(mod)} = \tilde{u}_j$, $\tilde{\tilde{u}}_j^{(mod)} = \tilde{\tilde{u}}_j$ in smooth regions.

Class 4.

Recall limiter. TVDM. TVBM

- MUSCL (minmod). $\Rightarrow$ Example 1
- bound . preserving $\Rightarrow$ Example 2.

③ Example 2.

I° Maximum principle.

$$m = \min_{x} u^0(x) \quad , \quad M = \max_{x} u^0(x).$$

$$m \le u(x,t) \le M \qquad \forall t, \forall x.$$

If $k=0$:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda\left(\hat{f}(\bar{u}_j^n, \bar{u}_{j+1}^n) - \hat{f}(\bar{u}_{j-1}^n, \bar{u}_j^n)\right) \qquad \hat{f}(\uparrow, \downarrow) \qquad \text{4-(1)}$$

$$= H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n)$$

then $H_\lambda(\uparrow, \uparrow, \uparrow)$ for $\lambda \le \lambda_0$.

$\hookrightarrow$ 求偏导. $\Rightarrow H_{\lambda_2} = 1 - \lambda \hat{f}_1 + \lambda \hat{f}_2 = 1 - \lambda(\overline{\hat{f}_1 - \hat{f}_2}) \geq 0 \Rightarrow \lambda \le \frac{1}{L}.$

$L \geqslant \cdot \geqslant 0$

Lipschitz constant

If $m \le \bar{u}_j^n \le M$. $\forall j$. then we got Maximum principle:

$$M. \ge \bar{u}_j^{n+1} = H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) \ge H_\lambda(m, m, m) = m$$

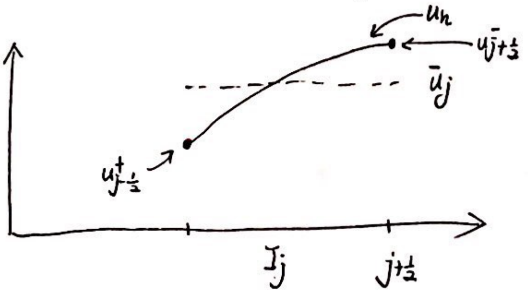
If $k>0$:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda\left(\hat{f}(u_{j+\frac12}^-, u_{j+\frac12}^+) - \hat{f}(u_{j-\frac12}^-, u_{j-\frac12}^+)\right) \qquad \text{4-(2)}$$

$$= G_\lambda(\bar{u}_j^n, u_{j+\frac12}^-, u_{j+\frac12}^+, u_{j-\frac12}^-, u_{j-\frac12}^+)$$

Hope.

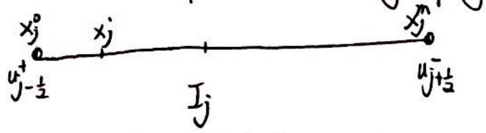
$$G_\lambda(\underset{\checkmark}{\uparrow}, \underset{\underset{\downarrow}{\times}}{\uparrow}, \underset{\checkmark}{\uparrow}, \underset{\checkmark}{\uparrow}, \underset{\underset{\downarrow}{\times}}{\uparrow}).$$



Recall that

m+1 points: exact for polys of degree $\le \underline{2m-1}$: $(1+m) + (m+1) - 2 = 2m$

↑ points ↑ weights ; boundary points fixed



Gauss – Lobatto points.

$$\Rightarrow \bar{u}_j = \sum_{l=0}^{m} w_l\, P(x_j^l) \quad (\text{exact}), \quad w_l > 0 \text{ and } \sum_{l=0}^{m} w_l = 1.$$

$$\text{then}\quad \bar u_j = \sum_{l=0}^{m} w_l\, p(x_j^l)$$

$$= w_0\, u^+_{j-\frac12}$$

$$+ w_m\, u^-_{j+\frac12}$$

$$+ \sum_{l=1}^{m-1} w_l\, p(x_j^l).$$

From (4-2), we have.

$$\bar u_j^{n+1} = w_0\Big[\underline{u^+_{j-\frac12}} - \frac{\lambda}{w_0}\big(\underline{\hat f(u^+_{j-\frac12},\ u^-_{j+\frac12})} - \hat f(u^-_{j-\frac12},\ \underline{u^+_{j-\frac12}})\big)\Big]$$

消去

$$+ w_m\Big[\underline{u^-_{j+\frac12}} - \frac{\lambda}{w_m}\big(\hat f(\underline{u^-_{j+\frac12}},\ u^+_{j+\frac12}) - \underline{\hat f(u^+_{j-\frac12},\ u^-_{j+\frac12})}\big)\Big]$$

$$+ \sum_{l=1}^{m-1} w_l\, p(x_j^l)$$

$$= w_0\, \underbrace{H_{\frac{\lambda}{w_0}}(u^-_{j-\frac12},\ u^+_{j-\frac12},\ u^-_{j+\frac12})}_{\in[m,M]} \quad \Rightarrow \lambda \le w_0\lambda_0$$

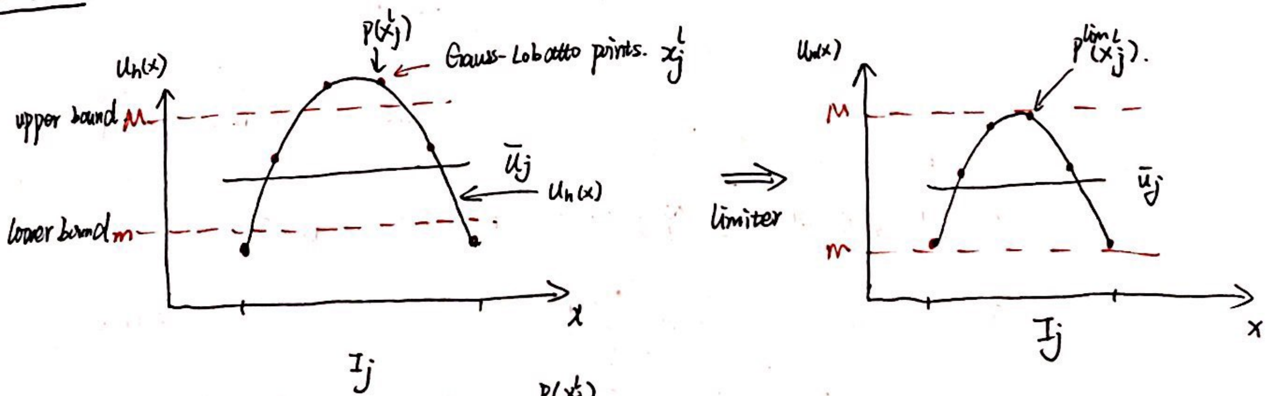
$$\updownarrow\ w_0 = w_m.$$

$$+ w_m\, \underbrace{H_{\frac{\lambda}{w_m}}(u^+_{j-\frac12},\ u^-_{j+\frac12},\ u^+_{j-\frac12})}_{\in[m,M)} \quad \Rightarrow \lambda \le w_m\lambda_0$$

$$+ \sum_{l=1}^{m-1} w_l\, \underbrace{p(x_j^l)}_{\text{suppose}.\in[m,M]}$$

If suppose (a) $m \le u_n^n(x_j^l) \le M$, where x_j^l is Gauss-Lobatto point, (b) $\lambda \le w_0\lambda_0$
then we have $m \le \bar u_j^{n+1} \le M$.

2°. Limiter



$p(x_j^l)$

引入 θ_j 压缩 u_h ⟹ Gauss-Lobatto points 的 value $\in[m,M]$
where $\theta_j(p(x) - \bar u_j) + \bar u_j$, $0 \le \theta_j \le 1$

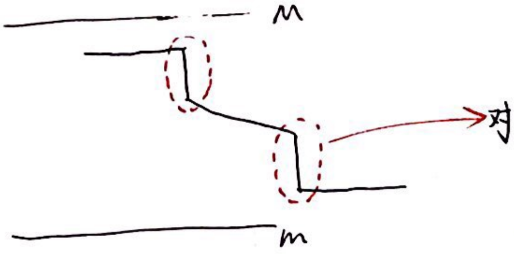
$$\text{Let}\ \begin{cases} M_j = \max\limits_{0\le l\le m} p(x_j^l), \\ m_j = \min\limits_{0\le l\le m} p(x_j^l). \end{cases} \qquad \text{Limiter}\Longrightarrow \left[\begin{array}{l} p^{lim}(x) = \theta_j(p(x) - \bar u_j) + \bar u_j \\ \quad 0 \le \theta_j \le 1 \end{array}\right]$$

$$\text{then}\quad \theta_j = \min\left\{1,\ \frac{M - \bar u_j}{M_j - \bar u_j},\ \frac{m - \bar u_j}{m_j - \bar u_j}\right\}.$$

3° Theorem If $|p(x) - u^{exact}(x)| \le Ch^{k+1}$,

then $|p^{lim}(x) - \underset{\uparrow \atop \text{smooth.}}{u^{exact}(x)}| \le C_1 h^{k+1}$.

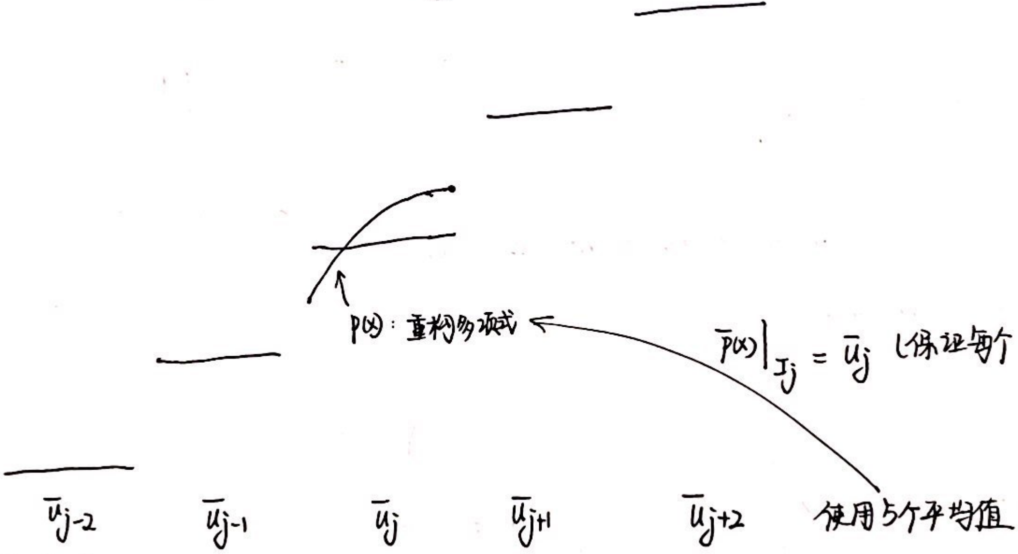
4° problem.



对 [m, M] 内部的跳跃无效.

④ Example 3. WENO Limiter.

(Weighted essentially non-oscillatory).



p(x): 重构多项式

$\overline{p(x)}\Big|_{I_j} = \bar u_j$ (保证每个 cell 中 mean value 不变)

使用5个平均值

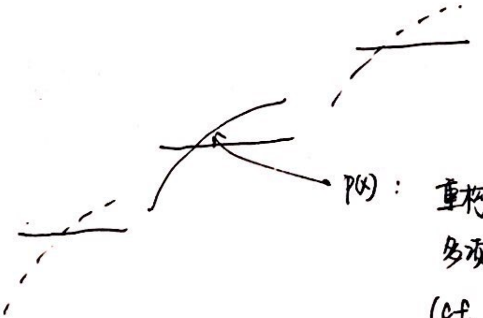
(cf. JX Qiu)

step 1. Troubled cell indicator

step 2. $u_h(x) \longleftarrow p(x)$ from WENO. (In troubled cell).

problem: 相邻 cell 使用过多, 影响 DG data structure.

$\Longrightarrow$



p(x): 重构时不仅使用平均值, 使用相邻cell 多项式更多 information

(cf. X. Zhang, J. Zhu).

III.

Consider $\begin{cases} u_t = u_{xx} \\ u(x,0) = u^0(x) \end{cases}$

1. Idea I (From conservation Law).

$$\Rightarrow \quad u_t + \underbrace{(-u_x)}_{f(u)}{}_x = 0$$

Recall: Find $u_h \in V_h$, s.t. $\forall v \in V_h$

$$\int_{I_j} (u_h)_t v\, dx - \int_{I_j} \underset{\substack{\downarrow \\ (-u_h)_x}}{\underline{f(u_h)}}\, v_x dx + \underset{\substack{\downarrow \\ \hat{u}_{x_{j+\frac12}}}}{\underline{\hat{f}_{j+\frac12}}}\, v^-_{j+\frac12} - \underset{\substack{\downarrow \\ \hat{u}_{x_{j-\frac12}}}}{\underline{\hat{f}_{j-\frac12}}}\, v^+_{j-\frac12} = 0$$

where $\quad \hat{f}_{j+\frac12} = \hat{f}\,(u^-_{j+\frac12}, u^+_{j+\frac12}),\ \hat{f}(\uparrow, \downarrow)$

Let $\quad \hat{u}_x = \hat{g}(u_{x^-_{j+\frac12}}, u_{x^+_{j-\frac12}})$

$$\hat{u}_{x_{j+\frac12}} = \frac12\left((u_h)_{x^-_{j+\frac12}} + (u_h)_{x^+_{j+\frac12}}\right)$$

$\Rightarrow$ Find $u_h \in V_h$, s.t. $\forall v \in V_h$

$$\int_{I_j} (u_h)_t v\, dx + \int_{I_j} u_{h_x} v_x\, dx + \hat{u}_{x_{j+\frac12}} v^-_{j+\frac12} - \hat{u}_{x_{j-\frac12}} v^+_{j-\frac12} = 0.$$

} "Bad Scheme" (No stability).

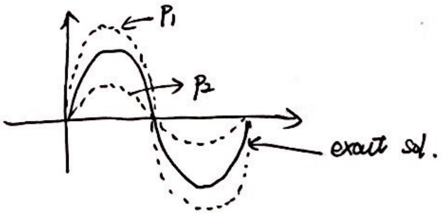
problem

If choose $u^0(x) = \sin x \Rightarrow u(x,t) = e^{-t}\sin x$.

Chose real solution ⇒ error rates

$P_1 \rightarrow 0$

$P_2 \rightarrow 0$ … 无 rates!



chose reference solution ⇒ error rates.

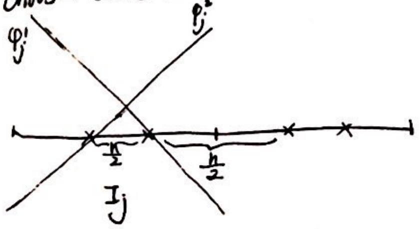
If $u_h = u + Ch^r$, $\Rightarrow u_{2h} = u + C(2h)^r$

$$\Rightarrow \quad u_{2h} - u_h = (u_{2h} - u) - (u_h - u) = C(2h)^r - Ch^r = C(1-2^r)h^r.$$

$P_1 \longrightarrow 1 \longrightarrow ?$ Q: 稳定, but 不相容?

Fact:

If we choose basis as



(Cf. M. zhang).

"稳定, 相容 ⇒ Convergence".

⇒ Fact: 相容 ⇒ 不稳定.

$u^{n+1} = G u^n$

$\|G^n\| \sim \frac{c}{h}$

2. Idea 2 : LDG (local DG).

① Scheme .

since $u_t = u_{xx}$: let $v = u_x$, $u_t = v_x$.

$$\Rightarrow \begin{cases} u_t - v_x = 0 \\ v - u_x = 0 \end{cases}$$

Find $u_h, v_h \in V_h$, s.t. $\forall w, z \in V_h$, we have

$$\begin{cases} \int_{I_j} (u_h)_t\, w\, dx + \int_{I_j} v_h w_x\, dx - \hat{v}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0 \\ \int_{I_j} \underset{\uparrow \text{ local.}}{v_h}\, z\, dx + \int_{I_j} u_h z_x\, dx - \hat{u}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} z^+_{j+\frac{1}{2}} = 0 \end{cases}$$

where central flux

$$\begin{cases} \hat{u}_{j+\frac{1}{2}} = \frac{1}{2} (u^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}) \\ \hat{v}_{j+\frac{1}{2}} = \frac{1}{2} (v^-_{j+\frac{1}{2}} + v^+_{j+\frac{1}{2}}). \end{cases}$$

Alternating flux

$$\begin{cases} \hat{u}_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}} \\ \hat{v}_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} \end{cases}$$

Bassi - Rebay

<u>Fact</u> :

$P_0, P_1 \longrightarrow$ order 1

$P_2, P_3 \longrightarrow$ order 3

② Stability .

Take $w = u_h \in V_h$, $z = v_h \in V_h$

$$\int_{I_j} (u_h)_t u_h\, dx + \int_{I_j} v_h (u_h)_x\, dx - \hat{v}_{j+\frac{1}{2}} (u_h)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} (u_h)^+_{j-\frac{1}{2}} = 0 \qquad (4\text{-}3)$$

$$+$$

$$\int_{I_j} v_h^2\, dx + \int_{I_j} u_h \cdot (v_h)_x\, dx - \hat{u}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} (v_h)^+_{j+\frac{1}{2}} = 0 \qquad (4\text{-}4)$$

$$= \int_{I_j} (u_h v_h)_x\, dx = u^-_{h\,j+\frac{1}{2}} v^-_{h\,j+\frac{1}{2}} - v^+_{h\,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \quad \circledast$$

$\sum_j [(4\text{-}3) + (4\text{-}4)] \Rightarrow$

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (u_h)^2\, dx + \int_0^1 (v_h)^2\, dx + \sum_j \textcircled{H}_{j-\frac{1}{2}} = 0.$$

where $\textcircled{H} = \underbrace{u^- v^- - u^+ v^+}_{\text{from } \circledast} - \hat{v} u^- + \hat{v} u^+ - \hat{u} v^- + \hat{u} v^+.$

$$\overset{\text{Alternating flux}}{=} u^- v^- - u^+ v^+ - v^+ u^- + v^+ u^+ - u^- v^- + u^- v^+ = 0.$$

↑

(central flux $\textcircled{H}$ still $= 0$).

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (u_h)^2 dx + \int_0^1 (v_h)^2 dx = 0$$

$$\Rightarrow \frac{1}{2}\|u_h(\cdot,t)\|^2 + \int_0^t \|v_h(\cdot,\tau)\|^2 d\tau = \frac{1}{2}\|u_h(\cdot,0)\|.$$

H.W. #4.

(1) Code the "bad" scheme for $\begin{cases} u_t = u_{xx} \\ u(x,0) = \sin x \end{cases}$ up to $t=1$.

Error table & picture for P^1 & P^2.

also "error table" for $u_h - u_{2h}$

(2) Same thing for LDG with central flux. (Error table)

(3) - - - - - - - - alternating flux.

Class 5

III Recall that $\begin{cases} u_t = u_{xx} \\ u(x,0) = u^0(x) \end{cases}$

2. Idea (LDG)

LDG: Find $u_h, v_h \in V_h$, s.t. $\forall w, z \in V_h$.

① scheme

$$(LDG)\quad \begin{cases} \int_{I_j} (u_h)_t\, w\,dx + \int_{I_j} v_h w_x\,dx - v^+_{j+\frac12} w^-_{j+\frac12} + v^+_{j-\frac12} w^+_{j-\frac12} = 0 \\ \int_{I_j} v_h z\,dx + \int_{I_j} u_h z_x\,dx - u^-_{j+\frac12} z^-_{j+\frac12} + u^-_{j-\frac12} z^+_{j-\frac12} = 0 \end{cases} \qquad (5\text{-}1)$$

② Stability $\quad \frac12 \frac{d}{dt}\int_0^1 (u_h)^2 dx + \int_0^1 (v_h)^2 dx = 0$

③ Error Estimate.

Let $e_u = u - u_h \quad e_v = v - v_h \quad (v = u_x)$

(1) u, and v satisfy the scheme (LDG).

(2) e_u, and e_v satisfy the scheme (LDG). $\Rightarrow$ error equations

(3) Let $e_u = (u - Pu) - (u_h - Pu) = \eta_u - \xi_u$

$e_v = (v - Qv) - (v_h - Qv) = \eta_v - \xi_v$.

(4) Take $w = \xi_u$, $z = \xi_v$. in error equations to get

$$\int_{I_j} (\xi_u)_t \xi_u\,dx + \int_{I_j} \xi_v (\xi_u)_x\,dx - \xi^+_{v_{j+\frac12}} \xi^-_{u_{j+\frac12}} + \xi^+_{v_{j-\frac12}} \xi^+_{u_{j-\frac12}} \qquad (5\text{-}2)$$

$$= \int_{I_j} (\eta_u)_t \xi_n\,dx + \overbrace{\cancel{\int_{I_j} \eta_v (\xi_u)_x\,dx}}^{\text{suppose } \int_{I_j}(v-Qv)z\,dx=0\ \forall z\in P^{k-1}(I_j)} \overbrace{- \cancel{\eta^+_{v_{j+\frac12}} \xi^-_{u_{j+\frac12}}} + \cancel{\eta^+_{v_{j-\frac12}} \xi^+_{u_{j-\frac12}}}}^{(b)\text{ suppose } (v-Qv)^+_{j+\frac12}=0\ \forall j}$$

$$\int_{I_j} \xi_v \cdot \xi_v\,dx + \int_{I_j} \xi_u (\xi_v)_x\,dx - \xi^-_{u_{j+\frac12}} \xi^-_{v_{j+\frac12}} + \xi^-_{u_{j-\frac12}} \xi^+_{v_{j-\frac12}}$$

$$= \int_{I_j} \eta_v \cdot \xi_v\,dx + \underbrace{\cancel{\int_{I_j} \eta_u (\xi_v)_x\,dx}}_{\text{suppose } \int_{I_j}(u-Pu)w\,dx=0,\ \forall w\in P^{k-1}(I_j)} \underbrace{- \cancel{\eta^-_{u_{j+\frac12}} \xi^-_{v_{j+\frac12}}} + \cancel{\eta^-_{u_{j-\frac12}} \xi^+_{v_{j-\frac12}}}}_{(a)\text{ suppose } (u-Pu)^-_{j+\frac12}=0\ \forall j} \qquad (5\text{-}3)$$

suppose :

$$(a)\ \begin{cases}(u-Pu)^{-}_{j+\frac{1}{2}}=0\ , & \forall j\\ \int_{I_j}(u-Pu)w\,dx=0\ , & \forall w\in P^{k}{}'(I_j)\end{cases}\ \Rightarrow\ \begin{cases}\|u-Pu\|\le Ch^{k+1}.\\ \|u_t-Pu_t\|\le Ch^{k+1}.\end{cases}\qquad (5.4)$$

$$(b)\ \begin{cases}(v-Qv)^{+}_{j-\frac{1}{2}}=0\ , & \forall j\\ \int_{I_j}(v-Qv)z\,dx=0. & \forall z\in P^{k+1}(I_j)\end{cases}\ \Rightarrow\ \begin{cases}\|\ \ -Q\ \ \|\le Ch^{k+1}.\\ \|\ \ -Qv_t\|\le Ch^{k+1}.\end{cases}$$

see P4-5 (eqn. 4.5).

then $$(LHS)\ \overset{\uparrow}{=}\ \frac{1}{2}\frac{d}{dt}\int_0^1(\xi_u)^2dx+\int_0^1(\xi_v)^2dx$$

$$=\int_0^1(\eta_u)_t\,\xi_u\,dx+\int_0^1\eta_v\,\xi_v\,dx\qquad (RHS)$$

$$\le\|(\eta_u)_t\|\,\|\xi_u\|+\|\eta_v\|\,\|\xi_v\|$$

$$\underset{(5.4)}{\le}Ch^{2k+2}+\frac{1}{2}\|\xi_u\|^2+Ch^{2k+2}+\frac{1}{2}\|\xi_v\|$$

then $$\frac{1}{2}\frac{d}{dt}\|\xi_u\|^2+\frac{1}{2}\|\xi_v\|^2\le Ch^{2k+2}+\frac{1}{2}\|\xi_u\|^2$$

By Gronwall's inequality, we have

$$\|\xi_u(\cdot,t)\|^2+\int_0^1\|\xi_v(\cdot,t)\|^2dt\ \le Ch^{2k+2}.$$

Hence $$\|e_u(\cdot,t)\|^2+\int_0^1\|e_v(\cdot,t)\|^2dt\ \le\ \underset{O(h^{2k+2})}{\|e_u(\cdot,0)\|^2}+Ch^{2k+2}$$

$$\le Ch^{2k+2}.$$

Remark

LDG method can be defined for nolinear parabolic or convection-diffusion equations:

$$u_t+f(u)_x=(a(u)u_x)_x\ ,\quad a(u)\ge 0$$

- stability ✓
- error estimate ✓

$$u_t = u_{xx}$$

3. Idea 2.

① problem

$$\forall v \in V_h: \int_{I_j} u_t v\,dx = \int_{I_j} u_{xx} v\,dx = -\int_{I_j} u_x v_x\,dx + u_{x_{j+\frac12}}^{\ominus} v^-_{j+\frac12} - u_{x_{j-\frac12}}^{\oplus} v^{\oplus}_{j-\frac12} \quad (5\text{-}6)$$

可省略. u is real solution.

不可省, $v \in V_h$.

$\Rightarrow$ Find $u_h \in V_h$

$$\int_{I_j} (u_h)_t v\,dx = -\int_{I_j} (u_h)_x v_x\,dx + (\widehat{u_{hx}})_{j+\frac12} v^-_{j+\frac12} - (\widehat{u_{hx}})_{j-\frac12} v^+_{j-\frac12}. \quad (5\text{-}7)$$

For stability estimate. choose $v = u_h$ in (5-7) to get.

$$LHS = \frac12 \frac{d}{dt} \int_{I_j} (u_h)^2 dx$$

$$RHS = -\int_{I_j} ((u_h)_x)^2 dx + \sum_j (\widehat{u_{hx}})_{j+\frac12} v^-_{j+\frac12} - \sum_j (\widehat{u_{hx}})_{\underline{j-\frac12}} \underline{v^+_{j-\frac12}}$$

$$= \underbrace{-\int_{I_j} (u_{hx})^2 dx}_{\text{help (放左边)}} \underbrace{-\sum_j \hat{u}_{n_{x_{j+\frac12}}} [u_n]_{j+\frac12}}_{\text{trouble}}$$

change index

where $[u]_{j+\frac12} = u^+_{j+\frac12} - u^-_{j+\frac12}$ (无符号!)

$\Rightarrow$ no stability.

② Approach I.

$\sum_j$ (5-7) and add new terms to get.

add new term.

(SIPG)
$$\sum_j \int_{I_j} (u_h)_t v\,dx = -\sum_j \int_{I_j} (u_h)_x v_x\,dx - \sum_j \Big[(\widehat{u_{hx}})_{j+\frac12} [v]_{j+\frac12} \underbrace{+ (\widehat{v_x})_{j+\frac12} [u_h]_{j+\frac12}}_{\text{对称项}} \underbrace{+ \frac{c}{h} [u_h]_{j+\frac12} [v]_{j+\frac12}}_{\text{internal penalty}} \Big]$$

- advantage: SIPG is stable.
- disadvantage: the choice of "c" in internal penalty (large enough).

③ Approach II.

$\sum_j$(5-7) and add new term to get

(NIPG) (Baumann-Oden method)
$$\sum \int_{I_j} (u_h)_t v\,dx = -\sum_j \int_{I_j} (u_h)_x v_x\,dx - \sum_j \Big[(\widehat{u_{hx}})_{j+\frac12} [v]_{j+\frac12} - (\widehat{v_x})_{j+\frac12} [u_h]_{j+\frac12} \Big]$$

- advantage: no penalty & c.
- disadvantage 偶数 掉一阶 (error estimate 掉阶)

④ Approach III

Ultra-Weak DG.

Find $u_h \in V_h$, s.t. $\forall\, v \in V_h$

(Ultra-Weak) DG

$$\int_{I_j} (u_h)_t\, v\, dx = \int_{I_j} u_h\, v_{xx}\, dx + \hat{u}_{x\,j+\frac{1}{2}}\, v^-_{j+\frac{1}{2}} - \hat{u}_{x\,j-\frac{1}{2}}\, v^+_{j-\frac{1}{2}}$$

$$- \hat{u}_{j+\frac{1}{2}}\, v^-_{x\,j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}\, v^+_{x\,j-\frac{1}{2}}.$$

If choose $\begin{cases} \hat{u}_{x\,j+\frac{1}{2}} = u^+_{hx\,j+\frac{1}{2}} + \underbrace{\frac{c}{h}[u_h]_{j+\frac{1}{2}}}_{\text{internal penalty term}} \\ \hat{u}_{j+\frac{1}{2}} = u^+_{h\,j+\frac{1}{2}} \end{cases}$ $\Rightarrow$ Stable scheme.

(cf. Y. Cheng)

IV. High order problem.

KdV equation

$$u_t + 6uu_x = \varepsilon u_{xxx}.$$

Dispersive wave equation

$$\begin{cases} u_t = u_{xxx} \\ u(x,0) = u^0(x) \end{cases}$$

Recall that $\begin{cases} u_t + u_x = 0 \\ u(x,0) = \sin x \end{cases} \Rightarrow u(x,t) = \sin(x-t).$ (convection)

$\begin{cases} u_t = u_{xx}. \\ u(x,0) = \sin x \end{cases} \Rightarrow u(x,t) = e^{-t}\sin x.$ (diffusion)

$\begin{cases} u_t = u_{xxx} \\ u(x,0) = u^0(x) = \sin x \end{cases} \Rightarrow u(x,t) = \sin(x-t)$

I. Idea I.

Now consider

① scheme. u, $v = u_x$, $w = v_x = u_{xx}$

$$\Rightarrow \begin{cases} u_t - w_x = 0 \\ w - v_x = 0 \\ v - u_x = 0 \end{cases}$$

$\Rightarrow$ Find $u_h, v_h, w_h \in V_h$, s.t. $\forall r, s, z \in V_h$.

$$\begin{cases} \int_{I_j} (u_h)_t r\,dx + \int_{I_j} w_h r_x\,dx - \hat{w}_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}} = 0 & (5\text{-}8) \\ \int_{I_j} (w_h)\cdot s\,dx + \int_{I_j} v_h\cdot s_x\,dx - \hat{v}_{j+\frac{1}{2}} s^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} s^+_{j-\frac{1}{2}} = 0 & (5\text{-}9) \\ \int_{I_j} v_h\cdot z\,dx + \int_{I_j} u_h\cdot z_x\,dx - \hat{u}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} = 0 & (5\text{-}10) \end{cases}$$

(cf. J. Yan).

$$\text{where}\quad \begin{array}{ccc} u & \overset{v}{u_x} & \overset{w}{u_{xx}} \\ \hat{u}=u^- & \hat{v}=v^- & \hat{w}=w^+ \end{array}$$

② Stability.

Choose $r = u_h$, $z = w_h$, $s = -v_h$.

$$\int_{I_j} (u_h)_t \cdot u_h\, dx + \int_{I_j} w_h\, u_{hx}\, dx - w^+_{h\,j+\frac{1}{2}}\, u^-_{h\,j+\frac{1}{2}} + w^+_{h\,j-\frac{1}{2}}\, u^+_{h\,j-\frac{1}{2}} = 0 \qquad (5\text{-}11)$$

$$\int_{I_j} v_h \cdot w_h\, dx + \int_{I_j} u_h \cdot (w_h)_x\, dx - u^-_{h\,j+\frac{1}{2}}\, w^-_{h\,j+\frac{1}{2}} + u^-_{h\,j-\frac{1}{2}}\, w^+_{h\,j-\frac{1}{2}} = 0 \qquad (5\text{-}12)$$

$$0 = \boxed{\int_{I_j} v_h \cdot w_h\, dx - \int_{I_j} w_h \cdot v_h\, dx}$$

$$-\int_{I_j} w_h \cdot v_h\, dx - \int_{I_j} v_h \cdot (v_h)_x\, dx + v^-_{h\,j+\frac{1}{2}}\, v^-_{h\,j+\frac{1}{2}} - v^-_{h\,j-\frac{1}{2}}\, v^+_{h\,j-\frac{1}{2}} = 0 \qquad (5\text{-}13)$$

$\sum_j [(5\text{-}11) + (5\text{-}12) + (5\text{-}13)]$ to get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_h)^2\, dx + \sum_j Ⓗ_{j-\frac{1}{2}} = 0$$

$$\text{where } Ⓗ = w^-u^- - w^+u^+ + \frac{1}{2}(v^-)^2 + \frac{1}{2}(v^+)^2$$

$$\boxed{-w^+u^- + w^+u^+ - u^-w^- + u^-w^+ + v^-v^- - v^-v^+}$$

$$= \frac{1}{2}(v^+ - v^-)^2 \geqslant 0$$

<u>Remark</u>: The scheme can be defined for quite general nolinear dispersive wave equations, with the same stability analysis.

③ Error Estimate.

(cf. J. Yan) → loss "little" error rate.

(cf. Y. Xu) → optimal error rate.

2. Idea 2. [Ultra-weak DG]

① scheme.

Find $u_h \in V_h$, s.t. $\forall\ v \in V_h$,

$$\int_{I_j} (u_h)_t v\,dx = -\int_{I_j} u_h v_{xxx}\,dx + \hat{u}_{xx_{j+\frac12}} v^-_{j+\frac12} - \hat{u}_{xx_{j-\frac12}} v^+_{j-\frac12} - \hat{u}_{x_{j+\frac12}} v^-_{x_{j+\frac12}} + \hat{u}_{x_{j-\frac12}} v^+_{x_{j-\frac12}} + \hat{u}_{j+\frac12} v^-_{xx_{j+\frac12}} - \hat{u}_{j-\frac12} v^+_{xx_{j-\frac12}} \qquad (5\text{-}14)$$

where.

u.	u_x	u_{xx}
$\hat{u} = u^-$	$\hat{u}_x = u_x^-$	$\hat{u}_{xx} = u_{xx}^+$.

② Stability.

Take $v = u_h$ in (5-14) to get.

$$\int_{I_j} (u_h)_t u_h\,dx = -\int_{I_j} u_h (u_h)_{xxx}\,dx + \boxed{\begin{aligned} & u^+_{hxx_{j+\frac12}} u^-_{h_{j+\frac12}} - u^+_{hxx_{j-\frac12}} u^+_{h_{j-\frac12}} \\ - & u^-_{hx_{j+\frac12}} u^-_{hx_{j+\frac12}} + u^-_{hx_{j-\frac12}} u^+_{hx_{j-\frac12}} \\ + & u^-_{h_{j+\frac12}} u^-_{hxx_{j+\frac12}} - u^-_{h_{j-\frac12}} u^+_{hxx_{j-\frac12}} \end{aligned}} \qquad (5.15)$$

since

$$\int_{I_j} u \cdot u_{xxx}\,dx = -\int_{I_j} u_x u_{xx}\,dx + u^- u^-_{xx_{j+\frac12}} - u^+ u^+_{xx_{j-\frac12}}$$

$$= -\frac12\left[(u^-_{x_{j+\frac12}})^2 - (u^+_{x_{j-\frac12}})^2 \right] + u^- u^-_{xx_{j+\frac12}} - u^+ u^+_{xx_{j-\frac12}}$$

then (5-15) $\Longrightarrow$

$$\int_{I_j} (u_h)_t u_h\,dx = \frac12 (u^-_{hx_{j+\frac12}})^2 - \frac12 (u^+_{hx_{j-\frac12}})^2 - u^-_{h_{j+\frac12}} u^-_{xx_{j+\frac12}} + u^+_{h_{j-\frac12}} u^+_{xx_{j-\frac12}} + \boxed{} \qquad (5\text{-}16)$$

$\sum_j$ (5-16) $\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_h)^2\,dx = \sum_j \textcircled{H}_{j-\frac{1}{2}}.$$

where
$$\textcircled{H} = -u^- u_{xx}^- + u^+ u_{xx}^+ + \frac{1}{2}(u_x^-)^2 - \frac{1}{2}(u_x^+)^2$$
$$+\boxed{u_{xx}^+ u^- - u_{xx}^+ u^+ - u_x^- u_x^- + u_x^- u_x^+ + u^- u_{xx}^- - u^- u_{xx}^+}$$
$$= -\frac{1}{2}(u_x^+ - u_x^-)^2 \le 0$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}\int_0^1 (u_h)^2\,dx \le 0$$

③ Error estimate.

$$e = u - u_h$$

(1) u satisfies the scheme

(2) e satisfies the scheme $\Rightarrow$ error equation

(3) $e = (u - Pu) - (u_h - Pu) = \eta + \xi.$

Take $v = \xi \in V_h$ in error equation. to get.

$$\int_{I_j} \xi_t \xi\,dx + \xi_{j+\frac{1}{2}}^- \xi_{xx\,j+\frac{1}{2}}^- - \xi_{j-\frac{1}{2}}^- \xi_{xx\,j-\frac{1}{2}}^+ + \frac{1}{2}(\xi_x^+)^2_{j-\frac{1}{2}} - \frac{1}{2}(\xi_x^-)^2_{j+\frac{1}{2}}$$
$$\boxed{\begin{aligned} &- \xi_{xx\,j+\frac{1}{2}}^+ \xi_{j+\frac{1}{2}}^- + \xi_{xx\,j-\frac{1}{2}}^+ \xi_{j-\frac{1}{2}}^+ + \xi_{x\,j+\frac{1}{2}}^- \xi_{x\,j+\frac{1}{2}}^- - \xi_{x\,j-\frac{1}{2}}^- \xi_{x\,j-\frac{1}{2}}^+ \\ &- \xi_{j+\frac{1}{2}}^- \xi_{xx\,j+\frac{1}{2}}^- + \xi_{j-\frac{1}{2}}^- \xi_{xx\,j-\frac{1}{2}}^+ \end{aligned}} \qquad (5\text{-}17)$$

$$= \int_{I_j} \eta_t \xi\,dx + \overbrace{\boxed{\int_{I_j} \eta\, \xi_{xxx}\,dx}}^{\text{suppose } \int_{I_j}(u-Pu)v\,dx=0,\ \forall v\in P^{k-3}(I_j)} + \overbrace{\cancel{\eta_{xx\,j+\frac{1}{2}}^+ \xi_{j+\frac{1}{2}}^-} - \cancel{\eta_{xx\,j-\frac{1}{2}}^+ \xi_{j-\frac{1}{2}}^+}}^{\text{suppose } (u_{xx}-(Pu)_{xx})^-_{j+\frac{1}{2}}=0,\ \forall j} \underbrace{- \cancel{\eta_{x\,j+\frac{1}{2}}^- \xi_{x\,j+\frac{1}{2}}^-} + \cancel{\eta_{x\,j-\frac{1}{2}}^- \xi_{x\,j-\frac{1}{2}}^+}}_{\substack{\text{suppose} \\ (u_x-(Pu)_x)^-_{j+\frac{1}{2}}=0\ \ \forall j}}$$
$$\underbrace{+ \cancel{\eta_{j+\frac{1}{2}}^- \xi_{xx\,j+\frac{1}{2}}^-} - \cancel{\eta_{j-\frac{1}{2}}^- \xi_{xx\,j-\frac{1}{2}}^+}}_{\text{suppose } (u-Pu)^-_{j+\frac{1}{2}}=0\ \ \forall j.}.$$

Suppose :

$$\begin{cases} (u-Pu)^-_{j+\frac{1}{2}}=0 & \forall j \\ (u_x-(Pu)_x)^-_{j+\frac{1}{2}}=0 & \forall j \\ (u_{xx}-(Pu)_{xx})^-_{j+\frac{1}{2}}=0 & \forall j \\ \int_{I_j}(u-Pu)v\,dx=0, & \forall v\in P^{\textcircled{k-3}}(I_j). \end{cases} \Longrightarrow \begin{cases} \|u-Pu\| \le Ch^{k+1} \\ \|u_t-Pu_t\| \le Ch^{k+1} \end{cases} \quad k\geqslant 2.$$

then $\sum_j (5\text{-}17)$ turns to

$$LHS = \frac{1}{2}\frac{d}{dt}\int_0^1 \xi^2\,dx - \underbrace{\sum_j \textcircled{H}_{j-\frac{1}{2}}}_{\geqslant 0} \leftarrow \text{(see P5-6. (5-11)-(5-13))}$$

$$= \int_0^1 \eta_t\,\xi\,dx \le \|\eta_t\|\,\|\xi\| \quad (RHS)$$

$$\Rightarrow \quad \frac{d}{dt}\|\xi\| \le C\|\eta_t\| \le Ch^{k+1}$$

$$\Rightarrow \quad \|\xi(\cdot,t)\| \le C(1+t)\,h^{k+1}$$

$$\Rightarrow \quad \|e(\cdot,t)\| \le C(1+t)\,h^{k+1}.$$

V. $u_t + u_{xxxxxx} = 0$