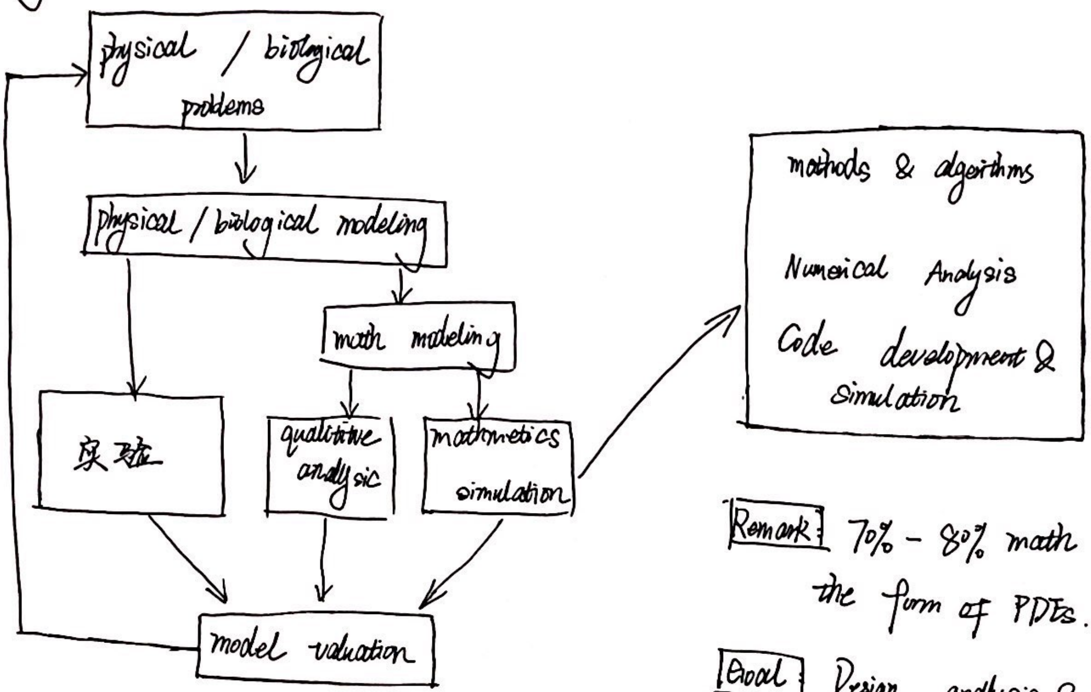
Finite Element Method and its Application

Class I.

FEM.

I. Big Picture.



$\boxed{\text{Remark}}$ 70% - 80% math models are given in the form of PDEs.

$\boxed{\text{Goal}}$ Design, analysis & Complementating of efficient & robust numerical <u>methods</u> (解离散化方程) & <u>Algorithms</u> for PDEs.
(离散化后解代数问题的 solver).

1. Source of PDEs.

① Descriptions of physical / biological "laws".

Example 1° [Newton's 2nd law of motions]

$$\underline{m\vec{a}} = \vec{F} \quad (\text{total force}).$$

Elasticity. $\vec{u}$ (displacement) $= \vec{u}(x,t)$ where $\sigma(\vec{u})$: stress tensor.

$$\underset{\text{density}}{\rho}\,\vec{u}_{tt} = \operatorname{div}(\underset{\text{internal force}}{\sigma(\vec{u})}) + \underset{\text{external force}}{\vec{f}}$$

Fluids. $\vec{v} = \vec{v}(x,t)$ (velocity)
$p = p(x,t)$ (pressure)

Navier-Stokes eqns.
$$\begin{cases} \rho \vec{v}_t = \operatorname{div}(\chi(\vec{v},p)) + \vec{f} \\ \operatorname{div} v = 0 \quad (\text{incompressibility}) \end{cases}$$

where
$$\begin{cases} \chi(\vec{v},p) = \nu\varepsilon(\vec{v}) + p\underset{\text{Identity matrix}}{I} \\ \underset{\text{deformation tensor}}{\varepsilon(\vec{v})} = \frac{1}{2}(\nabla\vec{v} + (\nabla\vec{v})^T) \end{cases}$$

Example 2° [Conservation Law]

$$\rho_t + \operatorname{div}(\underset{\text{velocity}}{\vec{v}}, \underset{\text{density}}{\rho}) = 0$$

$u_t + \operatorname{div} \vec{F} = 0$ (u : temperature)

$\uparrow$ flux

e.g. $\vec{F} = -D \nabla u$ (Fourier's law) D: conductivity $\to$ (constant) $(D>0)$

$\Rightarrow \quad u_t - D \Delta u = F$

If $u \simeq$ const in t, $u_t \approx 0$.

$\Rightarrow \quad -D\Delta u = F$

$\Rightarrow \quad -\Delta u = \frac{F}{D}$ (possion)

② Euler - Lagrange eqns. of Calculus of Variation.

<u>Main problem:</u>

Given "energy" functional

$$E(u) = \int_\Omega \underbrace{f(\nabla v, v, x)}_{\text{density function}} dx \qquad \Omega \subset \mathbb{R}^d \text{ open \& bounded.}$$

Find $u \in V$ s.t.

$$u = \underset{v \in V}{\operatorname{argmin}}\ E(v)$$

(i.e. $E(u) \leq E(v)$, $\forall v \in V$.)

<u>Fundamental Thm of CV.</u>

Idea: If $u \in \underset{v \in V}{\operatorname{argmin}}\ E(v)$, then $\left.\frac{\delta E(v)}{\delta v}\right|_{v=u} = 0$ (Necessary condition, Not sufficient.!) , where $\left.\frac{\delta E(v)}{\delta v}\right|_{v=u} = \lim_{t \to 0} \frac{E(u+tv) - E(u)}{t}$

(见补充I) Exercise 1° From $\delta E(v)/\delta v|_{v=u} = 0$, induce the following equation:

$$\Rightarrow \underbrace{\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(f_{p_j} (\nabla u, u, x) \right)}_{\operatorname{div}(L_p)} = f_u(\nabla u, u, x), \text{ where } f = f(p, v, x) \quad \circledast$$

Remark: $\circledast$ is a 2^{nd} order PDE for u (called E-L eqn).

Exercise 2°

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx \quad \left(\text{Dirichlet energy } f(\nabla v) = \tfrac{1}{2} |\nabla v|^2 \right)$$

$\Rightarrow$ E-L: $\Delta u = 0$
(Laplace eqn.)

$$E_p(v) = \frac{1}{p} \int_\Omega |\nabla v|^p dx, \text{ where } 1 \leq p < \infty. \quad \left(\text{P- Dirichlet energy, } f(\nabla v) = \tfrac{1}{p} |\nabla v|^p \right).$$

$\Rightarrow$ P-Laplace eqn.: $\Delta_p u = 0$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

Remark: $\begin{cases} p=2: & \Delta_p \text{ is a linear operator} \\ p>2: & \Delta_p \text{ is degenerate} \\ p<2: & \Delta_p \text{ is singular} \end{cases}$ If $p=1$: $\Delta_p u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ (1-Laplace operator)

③ "Non-physical" PDEs which do not belong to group ① & ②

Example: 1° Image Processing

2° Geometric Analysis.

3° Optimal control.

2. Solution Concepts.

① Classical solutions.

1°. pointwise concept, all derivatives in PDEs are <u>continuous</u> & PDEs are satisfied pathwise.

$$H(\nabla u, u, x) = 0$$

$u \in C^1(\Omega)$ & satisfies the PDE pointwise.

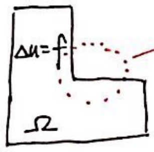
2°. <u>PDE theory</u>: developed in Hölder space $C^{k,\alpha}(\Omega)$ $(0<\alpha<1)$. (where $k=1$: 1th order eqn.
$k=2$: 2nd order eqn.)

Why Not use C^k, For $\Delta u = f$ in Ω → $A(x): D^2u + \vec{b}\cdot\nabla u + cu = f$

the reason is: $\exists f \in C^0 \not\Rightarrow u \in C^2$

but $\forall f \in C^{0,\alpha} \Rightarrow u \in C^{2,\alpha}$. (凸 domain).

Remark: classical solution may not exist even for very simple PDEs.



the domain 非凸. 无 classical solution. (见补充II)

② Strong Solutions.

1° PDE holds. a.e.

2° <u>PDE theory</u>: developed in Sobolev Spaces. $W^{k,p}(\Omega)$, (where $k \geq 0$, $k \leq p \leq \infty$)

Remark: All derivatives are <u>weak derivatives</u> in the PDE.

③ Weak Solutions / generalized solutions.

1°. Idea: Based on variational principles / weak formulations

Example: $E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v\, dx$

Find $u = \underset{v \in V}{\operatorname{argmin}}\, E(v)$, where $V = H_0^1(\Omega) = \{ v \in L^2(\Omega), |\nabla v| \in L^2(\Omega), v|_{\partial\Omega} = 0 \}$

<u>FTCV</u>: $\left.\frac{\delta E}{\delta v}\right|_{v=u} = 0$

G-derivative.

Define $\varphi(t) := E(u+tv)$, $\varphi(0)$ is the minimum of $\varphi(t)$, where $\varphi(t) = E(u+tv) = \frac{1}{2}\int_\Omega \underbrace{|\nabla u + t\nabla v|^2 dx}_{\text{quadratic poly of } t.}$

By Calculus fact, $\varphi'(0) = 0$

claim : (Exercise)

(a) Variational problem $\Rightarrow$ weak formulation

If $u \in \underset{v\in V}{\arg\min}\, E(v)$, then u satisfies

$$\Rightarrow \boxed{\overbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}^{a(u,v)} = \overbrace{\int_\Omega f \cdot v \, dx}^{F(v)} \qquad \forall v \in V} \quad \circledast$$

weak formulation / Variational principle.

(b) Weak formulation $\Rightarrow$

Integration by parts,

Assume $u \in H^2(u)$ (In order to use Integration by parts).

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \cdot v \, dx + \int_{\partial\Omega} \cancel{\frac{\partial u}{\partial n}} v \, dx \quad \left(\frac{\partial u}{\partial n} = 0\right)$$

$$= -\int_\Omega \Delta u \cdot v \, dx. \qquad \forall v \in V.$$

$$\Rightarrow \int_\Omega \underbrace{(\Delta u + f)}_{\in L^2(\Omega)} v \, dx = 0 \qquad \forall v \in V$$

$$\Rightarrow \Delta u + f = 0 \quad \text{in } L^2(\Omega)$$

Remark: Reverse the process, we obtain $\circledast$, this can be done for almost all PDEs.

Remark: Almost every PDE has a weak formulation $\circledast$ even if it may not correspond to a variational problem!

2° Definition

Weak Solutions are functions in V which satisfy $\circledast$ for given PDE.

Remark: key ingradients of weak solutions

(a). Looking for solutions in Sobolev Space which is weaker than strong solution Space.

(b) Satisfing an $\underbrace{\text{integral identity}}_{\circledast}$, not a differential equation.

Fact : If the weak solution are regular enough ($\underset{\text{strong}}{W^{k,p}(\Omega)}$ / $\underset{\text{classical}}{C^{k,\alpha}(\Omega)}$), they become strong or classical solution.

3° In general, a weak formulation of a linear PDE problem can be written (abstractly) as

[Galerkin] GP $\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = F(v) \end{cases}$

Remark: Different problems have different V, $a(\cdot,\cdot)$ and $F(\cdot)$, where

" Turn PDE problem into a functional problem "

$\begin{cases} V: \text{Hilbert Space or Banach Space} \\ a(\cdot,\cdot): \text{Bilinear form (i.e. linear each component) on } V * V;\ F(\cdot): \text{linear functional on } V. \end{cases}$

4° Existence & Uniqueness

<u>Lax-Milgram Theorem</u> (includes the existence of all linear elliptic problem)
[Sufficient condition]

Suppose V is a <u>Hilbert Space</u>, $a(\cdot,\cdot)$ & $F(\cdot)$ satisfy

(a) (continuity) $|a(w,v)| \leq C \|w\|_V \|v\|_V$, $\forall w, v \in V$ (where $C>0$, constant)

(b) (coercivity) $a(v,v) \geqslant \alpha \|v\|_V$ $\forall v \in V$ (where $\alpha > 0$, constant)

(c). F is bounded (or continuous).

Then (GP) has a unique solution $u \in V$.

Moreover, $\|u\|_V \leq \frac{\|F\|}{\alpha}$ ← functional norm.

<u>Exercise</u>: Read the proof of L-M Thm. (In particular, in the case $a(\cdot,\cdot)$ is symmetric).
i.e. $a(w,v) = a(v,w)$, $\forall v, w \in V$

Hint: $a(\cdot,\cdot)$ is an <u>inner product</u> on V.

5° Comparaision

Remark: Finite Difference Method

<u>Idea</u>: $-u_{xx} = f$, $0<x<1$

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u_{xx} + O(h^2)$$

↑ By Taylor expansion (suppose u is regular enough).

Approximate <u>classical derivatives</u> by difference quotients!

3. (Abstract) Galerkin Method

<u>Idea</u>
- Based on weak formulations.
- Approximate <u>infinite-dimensional</u> space V by finite-dimensional space V_N
  (Hilbert Space) (where $N = \dim V_N$)
  - Case I $V_N \subset V$ ⇒ Conforming Methods (协调元)
  - Case II $V_N \not\subset V$ ⇒ Non-Conforming Methods (非协调元).

<u>Galerkin Method</u>:

$$(GP)_N \begin{cases} \text{Find } u_N \in V_N \text{ s.t.} \\ a(u_N, v_N) = F(v_N) \quad \forall v_N \in V_N. \end{cases}$$

Lax-Milgram Theorem (In particular, in the case $a(\cdot,\cdot)$ is symmetric, then $a(\cdot,\cdot)$ is an inner product on V)

proof: For each fixed element $u \in V$, the mapping $v \to a(u,v)$ is a bounded linear functional on V where the Riesz Representation Theorem asserts the existence of a unique element $w \in V$ satisfying

$$a(u,v) = (w,v) \quad \forall v \in V \qquad (1)$$

Write $Au = w$ whenever (1) holds, then

$$a(u,v) = (Au,v) \quad \forall u, v \in V \qquad (2)$$

We first claim $A: V \to V$ is a bounded linear operator. (3)

If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in V$, for each $v \in V$, we have

$$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = a((\lambda_1 u_1 + \lambda_2 u_2), v) = \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v)$$
$$= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) = (\lambda_1 Au_1, v) + (\lambda_2 Au_2, v), \forall v \in V.$$

$\Rightarrow$ A is linear.

Furthermore, $\|Au\|^2 = (Au, Au) = a(u, Au) \le C\|u\|\|Au\|$

$\Rightarrow$ $\|Au\| \le C\|u\|$, so A is bounded.

Next we assert A is one-to-one, and the range $R(A)$ of A is closed in V. (4)

since $\alpha\|u\|^2 \le a(u,u) = (Au, u) \le \|Au\|\|u\|$

we have $\alpha\|u\| \le \|Au\|$. Hence A is 1-1, and $R(A) \overset{\text{closed}}{\subset} V$.

We demonstrate now $R(A) = V$. (5)

If not, since $R(A)$ is closed, there $\exists$ nonzero element $w \in V$ with $w \in R(A)^\perp$.

But this fact in turn implies the contradiction

$$\alpha\|w\|^2 \le a(w,w) = (Aw, w) = 0.$$

Observe once more from the Riesz Representation Theorem that

$$\langle F, v \rangle = (w, v) \quad \text{for all } v \in V. \qquad (6)$$

for some element $w \in V$.

By (4) and (5), we can find $u \in V$ satisfying $Au = w$. Then.

$$a(u,v) = (Au, v) = (w, v) = \langle F, v \rangle, \forall v \in V.$$

and this proves the existence.

Finally, we know there is at most one element $u \in V$.

If $\exists u$ & $\tilde{u}$ s.t. $\begin{cases} a(u,v) = \langle F, v \rangle \\ a(\tilde{u}, v) = \langle F, v \rangle \end{cases}$, then $a(u - \tilde{u}, v) = 0$ ($\forall v \in V$)

set $v = u - \tilde{u}$ to find $\alpha\|u - \tilde{u}\|^2 \le a(u-\tilde{u}, u-\tilde{u}) = 0$.

[补充 I]

1. 非线性下 Gateaux 的弱微分 $DF(x_0;h)=\lim_{t\to 0}\frac{f(x_0+th)-f(x_0)}{t}$ 1.

Frechet 的强微分 $F(x_0+h)-F(x_0)=Ah+\omega(x_0,h)$, where $\omega(x_0,h)=O(\|h\|)$

即 $\lim_{h\to 0}\frac{\|\omega(x_0,h)\|}{\|h\|}=0$

2. Frechet 可微强:

G可微 在 x_0 处且连续，则在 x_0 处 F 可微.

Exercise 1°:

Example: $I=\int_\Omega L(x,u,\nabla u)\,dx$.

$I(x)=\frac{1}{2}x^TAx \Rightarrow I(u+v)-I(u)\approx \boxed{I'(u)}v$

$I(x+\Delta x)-I(x)=\frac{1}{2}(x^T+\Delta x^T)A(x+\Delta x)-\frac{1}{2}x^TAx \Rightarrow I'(x)(h)=\frac{1}{2}(A^Tx,h)+\frac{1}{2}(Ax,h)$

$\approx \frac{1}{2}x^TA\Delta x+\frac{1}{2}\Delta x^TAx+\cdots$

$=\frac{1}{2}(A^Tx,\Delta x)+\frac{1}{2}(Ax,\Delta x)$

$I(u+v)-I(u)\approx I'(u)\,v$

$=\int_\Omega L(x,u+v,\nabla u+\nabla v)-L(x,u,\nabla u)\,dx \Rightarrow I'(u)(h)=\int_\Omega (L_u-\operatorname{div}(L_p))h\,dx$

$\approx\int_\Omega L_uv+L_p\cdot\nabla v\,dx \quad (v|_{\partial\Omega}=0)$

$=\int_\Omega (L_u-\operatorname{div}(L_p))v\,dx$

Exercise 2°: ① If $I(u)=\int_\Omega L(x,u,\nabla u)\,dx$, and $L(x,u,\nabla u)=\frac{1}{2}|\nabla u|^2=\frac{1}{2}p^2$, $p=\nabla u$.

$\Rightarrow L_u=0 \quad L_p=p$

$\Rightarrow L_u-\operatorname{div}(L_p)=-\operatorname{div}(p)=-\operatorname{div}(\nabla u)=0\Rightarrow \Delta u=0.$

② If $L(x,u,\nabla u)=\frac{1}{q}|\nabla u|^q=\frac{1}{q}|p|^q$, $p=\nabla u$.

then $L_u-\operatorname{div}(L_p)=-\operatorname{div}(|p|^{q-2}p)$

$=-\operatorname{div}(|\nabla u|^{q-2}\nabla u)$

where $(|p|^q)'=((|p|^2)^{\frac{q}{2}})'$

$=\frac{q}{2}(|p|^2)^{\frac{q}{2}-1}\cdot 2p$

$=q|p|^{q-2}p$

$\Rightarrow$ let $I'(u)=0 \Rightarrow \operatorname{div}(|\nabla u|^{q-2}\nabla u)=0\Rightarrow \Delta_p u=0.$

③ If $q=2$. then $\Delta u=0$.

[补充 II]

$\|u_h-u\|\le Ch^2$ 其中 C 与 H^2 有关.

Class 2.

Exercise: Derive weak formulation for

① $\begin{cases} -\operatorname{div}(a(x)\nabla u) + c(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

② $\begin{cases} -\operatorname{div}(a(x)\nabla u) + \vec{b(x)}\cdot\nabla u + c(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

③ $\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$

④ $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

Remark: $V_g = \{ v \in H^1(\Omega),\ v|_{\partial\Omega} = g \}$

V_g is NOT a vector space!

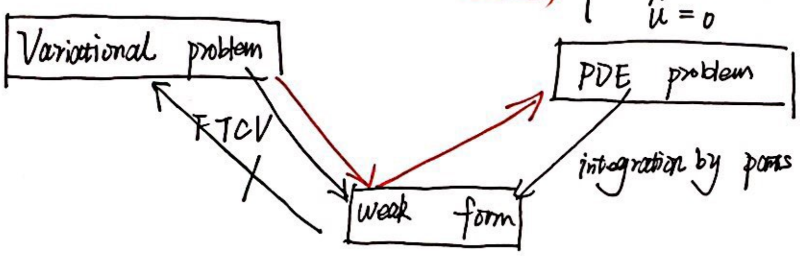
分析时.
先齐次化.
(与计算时技巧不同)

For analysis only, chose $w \in H^1(\Omega)$ such that $w|_{\partial\Omega} = g$

define $\hat{u} := u - w$ ($u = \hat{u} + w$).

Then $\begin{cases} -\Delta\hat{u} = f + \Delta w & \text{in } \Omega \\ \hat{u} = 0 & \text{on } \partial\Omega \end{cases}$

Recall:



$\rightarrow$ (red): Variational problem always corresponds to a PDE problem

$\rightarrow$ (black): The inverse not always holds.

Q: When does a PDE problem has an equivalent variational problem?

Theorem (GP) is equivalent the following Ritz problem

$$(RP) \quad u = \underset{v\in V}{\operatorname{argmin}}\, J(v), \quad \frac{1}{2}a(v,v) - F(v) = J(v).$$

iff $a(\cdot,\cdot)$ is symmetric.

proof: (GP) $\Longrightarrow$ (RP)

Want to prove $J(u) \le J(w)$ $\forall v \in V$

set $u = w - v$ or $w = u + v$

$$\begin{aligned} J(w) = J(u+v) &= \frac{1}{2}a(u+v, u+v) - \underset{\text{linear functional}}{F}(u+v) \\ &= \frac{1}{2}a(u,u) + \frac{1}{2}a(u,v) + \frac{1}{2}a(v,u) + \frac{1}{2}a(v,v) - F(u) - F(v) \\ &= J(u) + \underbrace{(a(u,v) - F(v))}_{=0} + \frac{1}{2}a(v,v) \\ &= J(u) + \underbrace{\frac{1}{2}a(v,v)}_{\ge 0} \\ &\ge J(u) \end{aligned}$$

(RP) $\Rightarrow$ (GP)

Let $\varphi(t) = J(u+tv)$. since J takes minimum at u, then $\varphi(0)$ is a minimum of φ.

then $\varphi'(0) = 0 \iff$ (GP)

3. (Abstract) Galerkin Method.

$$(GP)\ \begin{cases} \text{Find } u \in V \text{ s.t.} \\ a(u,v) = F(v) \quad \forall v \in V \end{cases}$$

Let $V_N \subset V$ ($N = \dim V_N$)

$$(GP)_N\ \begin{cases} \text{Find } u_N \in V_N, \text{ s.t.} \\ a(u_N, v_N) = F(v_N) \quad \forall v_N \in V_N \end{cases}$$

Q1: Existence & Uniqueness.

Theorem Suppose $a(\cdot,\cdot)$ & $F(\cdot)$ satisfy (a)–(c) from Lax-Milgram Theorem, Then $(GP)_N$ has a unique solution.

proof: Idea step1. Reduce the problem into a equivalently linear system problem
$Ax = b$.

Let $\{\varphi_j\}_{j=1}^N$ be basis for V_N, then $u_N(x) = \sum_{j=1}^N \xi_j \varphi_j(x)$
(NOT Unique!)

put $u_N(x)$ into $(GP)_N$, we got

$$a(u_N, v_N) = F(v_N)$$

$$\Rightarrow \sum_{j=1}^N \xi_j a(\varphi_j, v_N) = F(v_N)$$

$$\iff \sum_{j=1}^N \xi_j a(\varphi_j, \varphi_i) = F(\varphi_i) \qquad i = 1, 2, \dots N$$

$$\iff Ax = b$$

where $\begin{cases} A = [a_{ij}]_{i,j=1}^N, & a_{ij} = a(\varphi_j, \varphi_i). \\ x = [\xi_1, \xi_2, \dots \xi_n]^T. \\ b = [b_i]_{i=1}^N, & b_i = F(\varphi_i) \end{cases}$

step2.

Now we need to show A is non-singular (i.e. A^{-1} exists).

$Ax=0$ 只有0解. i.e. 解若存在一定唯一.

Exercise: prove that A is positive definite (and symmetric if is symmetric.)

Q2: Convergence & rate of Convergence

Theorem [Céa Lemma]

$$\|u-u_N\|_V \le \underset{\text{from (b) in L-M Thm.}}{\overset{\text{from (a) in L-M Thm}}{\frac{C}{\alpha}}} \inf_{\forall v_N \in V_N} \|u-v_N\|_V$$

proof: since $a(u, v_N) = F(v_N)$, $v_N \in \underline{V_N \subset V}$

then $\boxed{a(u-u_N, v_N)=0, \quad \forall v_N \in V_N}$ [Error equation]

$\Rightarrow (u-u_N) \perp V_N$ in $a(\cdot,\cdot)$ - "inner product"

} Galerkin Orthogonality

since $\alpha\|u-u_N\|_V^2 \le a(u-u_N, u-u_N) = a(u-u_N, u) - a(u-u_N, u_N)$

$$= a(u-u_N, u) - \underset{0}{a(u-u_N, u_N)} - \underset{0}{a(u-u_N, v_N)}$$

$$= a(u-u_N, u-v_N)$$

$$\le C\|u-u_N\|_V \|u-v_N\|_V.$$

then $\|u-u_N\|_V \le \frac{C}{\alpha}\|u-v_N\|_V$, $\forall v_N \in V_N$.

Remark 1. Céa Lemma reduces error estimate into a space approximation Problem which has been well studied long before FEM.

Remark 2. $\|u-u_N\|_V \le \frac{C}{\alpha}\|u-I_N u\|_V$

OR

$\|u-u_N\|_V \le \frac{C}{\alpha}\|u-Q_N u\|_V$

I_N: "interpolation" operator (into V_N)

Q_N: "projection" operator (into V_N)

Corollary: If $\lim_{N\to\infty} V_N = V$ (in some sense)

then $\lim_{N\to\infty} \|u-u_N\|_V = 0$.

Remark 3: In PDE analysis, $V_N = \text{span}\{\psi_1, \psi_2, \dots, \psi_N\}$

e.g. $\Delta\psi_j = \lambda_j \psi_j$, $j = 0, 1, 2, \dots$

Galerkin Method can be used to prove existence of solution for linear PDE problems. (see PDE book by L.C. Evans).

Remark 4: $V_N = \overset{\text{有限元空间}}{P_N(\Omega)} = \{$ set of polynomials whose total degree $\le N\}$ (整体多项式 / 全局多项式)

then Galerkin method is called spectral method.

Exercise: Suppose $a(\cdot,\cdot)$ is symmetric, prove that

$$\|u-u_N\|_V \le \sqrt{\frac{C}{\alpha}} \inf_{v_N \in V_N} \|u-v_N\|$$

II Finite element method. (Goal: to construct V_N) $\to V_h$

<u>Remark</u> FEM is a special Galerkin method (so are DG & spectral methods).

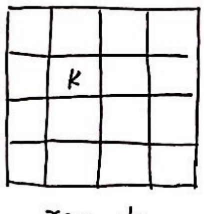
1. Building Blocks of FEM

① Mesh (triangulation)

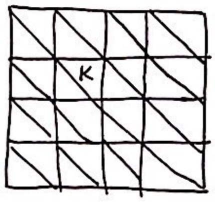
$\mathcal{T}_h = \{K\}$

where K is elements.

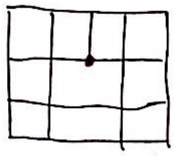
- Conforming Mesh.



rectangle



triangle



Non-Conforming

② $V_r(K) := V_h|_K$ is a polynomial space on K.

the order of polynomial

i.e. V_h consists of piecewise polynomials.

$$V_r(K) = \begin{cases} P_r(K) & \text{if } \mathcal{T}_h \text{ is a triangular mesh} \\ Q_r(K) & \text{if } \mathcal{T}_h \text{ is a rectangular mesh} \end{cases}$$

where $\begin{cases} P_r(K) = \{\text{set of polynomials whose total degree} \le r\} \\ Q_r(K) = \{\text{set of polynomials whose degree in each } x_i \le r\} \end{cases}$

<u>Fact</u>: $\begin{cases} \dim(P_r(K)) = \begin{pmatrix} d+r \\ r \end{pmatrix} = \dfrac{(d+r)!}{d!\, r!} \\ \dim(Q_r(K)) = (r+1)^d \end{cases}$ $\longleftarrow$

若 $x_1^{r_1} x_2^{r_2} \cdots x_d^{r_d}$, $r_1+r_2+\cdots+r_d = r$
then 共有 C^r_{d+r-1} 种组合
即共有 $C^0_{d-1} + C^1_d + C^2_{d+1} + \cdots + C^r_{d+r-1}$
$= C^0_d$
$= C^1_{d+1} + \cdots$
$= C^2_{d+2} + \cdots$
$= \cdots$
$= C^r_{d+r}$

<u>Goal</u> Construct $V_h \subset V$

- <u>Remark</u> V depends on the order of the underlying PDE.

Only need to consider $\underbrace{V = H^1(\Omega)}_{\text{for } 2^{nd} \text{ order elliptic PDE}}$ & $\underbrace{V = H^2(\Omega)}_{\text{for } 4^{th} \text{ order elliptic PDEs.}}$

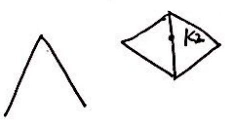
<u>Model PDE problem</u>

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

<u>Theorem</u> (a) $V_h \subset H^1(\Omega)$ if and only if $V_h \subset C^0(\Omega)$.

(b) $V_h \subset H^2(\Omega)$ if and only if $V_h \subset C^1(\Omega)$.

(see Ciarlet's book)



in $C^0(\Omega)$ but not in $C^1(\Omega)$

Now, consider $V = H^1(\Omega)$

$$V_h = \text{span}\{\underbrace{\varphi_1, \varphi_2, \dots \varphi_N}_{\text{form a basis for } V_h}\}$$

Nodal basis (a very special basis for V_h)

(Consider $r=1$, triangular mesh)

Let $\{p_j\}_{j=1}^{J_h}$ $\rightarrow$ ~~Note that~~ $N_h = J_h$ denote the labeling of all vertices of $\mathcal{T}_h$.

Define

$$\varphi_j(p_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad j = 1, 2, \dots J_h$$

Claim (a) $\{\varphi_j\}_{j=1}^{J_h}$ is well-defined.

(b) $\{\varphi_j\}_{j=1}^{J_h}$ is linearly independent.

Hence $\{\varphi_j\}_{j=1}^{J_h}$ is a basis for V_h.

(c) $\forall v_h \in V_h$, $v_h = \sum_{j=1}^{J_h} c_j \varphi_j$

then $c_j = v_h(p_j)$ $\quad j = 1, 2, \dots J_h$

This is the reason why $\{\varphi_j\}$ is called the nodal basis.

Remark Let u_h be the FE solution, and $u_h = \sum_{j=1}^{J_h} \xi_j \varphi_j$, then $Ax = b$

where $x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{J_h} \end{bmatrix} = \begin{bmatrix} u_h(p_1) \\ u_h(p_2) \\ \vdots \\ u_h(p_{J_h}) \end{bmatrix}$

系数恰好为有限元解在网点的值

Recall $A = [a_{ij}]$ $\qquad b = [b_i]$

$a_{ij} = a(\varphi_j, \varphi_i)$ $\qquad b = F(\varphi_i)$

Suppose $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ (poisson)

$$= \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \nabla(u|_K) \cdot \nabla(v|_K) \, dx$$

$$= \sum_{K \in \mathcal{T}_h} a_K(u|_K, v|_K)$$

Then $a_{ij} = a(\varphi_j, \varphi_i)$

$$= \sum_{K \in \mathcal{T}_h} a_K(\varphi_j|_K, \varphi_i|_K)$$

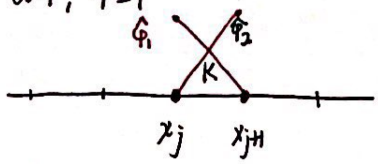
hence we only need to know expression/formula of $\varphi_j|_K$, which must be a polynomial!

Remark

Non-zero functions from $\{\varphi_j|_K\}_{j=1}^{J_h}$ forms a basis for $V_r(K)$

Examples of FE spaces

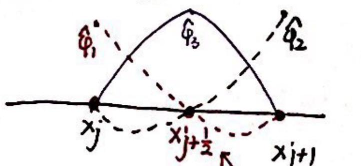
① $d=1$, $r=1$



自由度 $\dim(\underset{\substack{\uparrow \\ r=1}}{V_1}(K)) = \binom{d+r}{d} = \frac{(d+r)!}{d!\,r!} = \frac{2!}{1!\,1!} = 2$

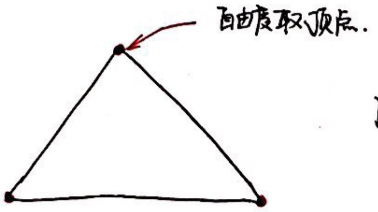
$$\begin{cases} \hat\varphi_1(x) = \dfrac{x - x_{j+1}}{x_j - x_{j+1}} \\ \hat\varphi_2(x) = \dfrac{x - x_j}{x_{j+1} - x_j} \end{cases}$$

② $d=1$, $r=2$



$\dim(V_2(K)) = 3$ 引入中点.

③ $d=2$, $r=1$



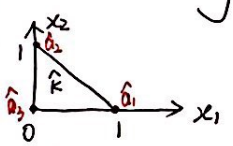
Degree of freedom (DOFs)

$v(x) = a + bx_1 + cx_2$

since $\dim(V_1(K)) = 3$ (a, b, c 三个自由度 (3个未知数))

Unisolvability: determined by DOFs

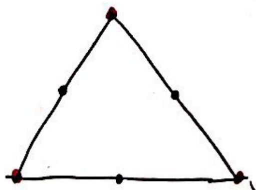
Remark: If



(标准单元)

then $\begin{cases} \hat\varphi_1(x) = x_1 \\ \hat\varphi_2(x) = x_2 \\ \hat\varphi_3(x) = 1 - x_1 - x_2 \end{cases}$

④ $d=2$, $r=2$



→ 顶点偏导连续性要求.

$\dim(V_2(K)) = \frac{4!}{2!\,2!} = 6 = \text{DOFs}$

下午习题补充

Exercise 1. Suppose $a(\cdot,\cdot)$ is symmetric, prove that

$$\|u-u_N\|_V \le \sqrt{\frac{C}{\alpha}}\inf_{v_N\in V_N}\|u-v_N\|$$

proof: Define $a(u,v) \triangleq \langle u,v\rangle$ ← if $a(\cdot,\cdot)$ is symmetric, then $a(\cdot,\cdot)$ can be regarded as a inner product.

Since $a(u-u_N, v_N)=0$, $\forall v_N\in V_N$.

i.e. $\langle u-u_N, v_N\rangle = 0 \quad \forall v_N\in V_N$

since $\langle u-u_N, u-u_N\rangle = \langle u-u_N, u\rangle - \langle u-u_N, \overset{0}{u_N}\rangle - \langle u-u_N, \overset{0}{v_N}\rangle = \langle u-u_N, u-v_N\rangle \le \|u-u_N\|_a\|u-v_N\|_a$

we have $\|u-u_N\|_a \le \inf_{v_N\in V_N}\|u-v_N\|_a$

norm induced by $a(\cdot,\cdot)$

since $a(u,u)\ge\alpha\|u\|_V^2$

then $\|u-u_N\|_a = \sqrt{a(u-u_N, u-u_N)} \ge \sqrt{\alpha}\|u-u_N\|_V$.

since $a(u,u)\le C\|u\|_V^2$

then $\|u-v_N\|_a = \sqrt{a(u-v_N, u-v_N)} \le \sqrt{C}\|u-v_N\|_V$

Hence

$$\|u-u_N\|_V \le \sqrt{\frac{C}{\alpha}}\inf_{v_N\in V_N}\|u-v_N\|_V.$$

2. Derive weak formulation.

Recall that

- Gauss-Green Theorem: suppose $u\in C^1(\bar U)$, then

$$\int_U u_{x_i}\,dx = \int_{\partial U} u n^i\,dS \quad (i=1,2,\dots n). \qquad (1)$$

n^i → n 的第i个分量. $(n=(n_1,n_2,\dots,n_i,\dots,n_n))$

- 散度定理:

$$\int_\Omega \mathrm{div}\,\vec u\,dx = \int_{\partial\Omega}\vec u\cdot n\,dS \qquad (2)$$

If $\vec u=(0,0,\dots \underset{\uparrow\, i\text{-th 分量}}{u}, 0\dots)$, then

- Integration-by-parts formula.

Let $u,v\in C^1(\bar U)$, then

$$\int_U u_{x_i} v\,dx = -\int_U u v_{x_i}\,dx + \int_{\partial U} u v n^i\,dS \quad (i=1,2,\dots,n) \qquad (3)$$

proof: Apply (1) to uv

- Green's formula. Let $u,v\in C^2(\bar U)$, then

(a) $\int_U \Delta u\,dx = \int_{\partial U}\frac{\partial u}{\partial n}\,dS$.

(b) $\int_U Dv\cdot Du\,dx = -\int_U u\Delta v\,dx + \int_{\partial U}\frac{\partial v}{\partial n}u\,dS$. (Let $\Delta v = \mathrm{div}(\nabla v)$)

(c) $\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,dS$.

proof: For (a), using (3) with u_{x_i} in place of u and $v\equiv 1$ / or use ∇u in place of $\vec u$ in (2) where we use $\frac{\partial u}{\partial n} = \nabla u\cdot n$

For (b), using (3) with $v=u_{x_i}$

For (c), write (b) with u & v interchanged and then subtract.

① $\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$

since $\begin{cases} u \in U \\ v \in V \end{cases}$, here $U = V$, hence $v = \frac{\partial v}{\partial n} = 0$ on $\partial\Omega$.

Using Δu in place of u in (c) to get $\int_U \Delta u \Delta v - v \Delta^2 u \, dx = \int_{\partial U} \Delta u \underset{0}{\underset{\|}{\frac{\partial v}{\partial n}}} - \underset{0}{\underset{\|}{v}} \frac{\partial \Delta u}{\partial n} \, dS = 0$

$$\Rightarrow \int \Delta^2 u \, v \, dx = \int_U \Delta u \cdot \Delta v \, dx = a(u, v).$$

Then the weak formulation is $\int \Delta u \cdot \Delta v \, dx = \int f \cdot v \, dx$.

1° "positive operator" 2° 为保证正定性, $c(x) \geq 0$

② $\begin{cases} -\operatorname{div}(a(x) \nabla u) + \vec{b}(x) \cdot \nabla u + c(x) u = f & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases}$

1° For "Δ", consider $\Delta u = \lambda u$, then $\int \Delta u \cdot u \, dx = \lambda \int u \cdot u \, dx \Rightarrow -\int |\nabla u|^2 dx + \int_{\partial U} \frac{\partial u}{\partial n} \cdot d = \lambda \int |u|^2 dx$

then Δ 为负 Laplace operator. $\Rightarrow \lambda = -\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} \leq 0$

Hence $-\Delta$ 为 positive operator.

2° positive operator $+ CI$ is still positive if $C \geq 0$.

3° 引入 $\vec{b}(x)$ 之后, 通常会使双线性泛函失去正定性, 从而影响解的唯一性.
因而可通过引入对 b 与 $c(x)$ 的要求, 使双线性泛函恢复其正定性.

Weak formulation:

Consider $(Lu, v) = (u, L^* v)$ if $L = \operatorname{div}$, $L^* = \nabla$, i.e. $(\operatorname{div} \vec{u}, v) = (\vec{u}, \nabla v)$

Hence the weak formulation for problem ② is

$$\begin{cases} B[u, v] = \int a(x) \nabla u \cdot \nabla v \, dx + \int \vec{b}(x) \cdot \nabla u \cdot v \, dx + \int c(x) u \cdot v \, dx \\ F[v] = \int f \cdot v \, dx. \end{cases}$$

For $B[u, v]$, 因为无正定性, 不能再使用 M-L Thm 证明其存在性.

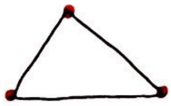
Class 3.

2. Triangular

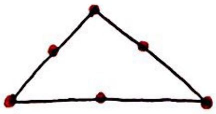
① Triangular H^1 – FEs

Lagrange H^1- FEs

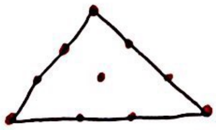
Example : (a) $d=2$, $r=1$ (b) $d=2$, $r=2$ (c). $d=2$, $r=3$



$\dim(P_1(K))=3$



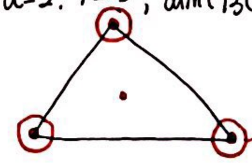
$\dim(P_2(K))=6$



$\dim(P_3(K))=\frac{(d+r)!}{d!\,r!}=10$

Remark: Hermite H^1- FEs

(d.) $d=2$, $r=3$, $\dim(P_3(K))=10$



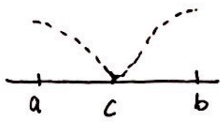
1st-order directional derivative
$d=2$. number $=2$.

② Example of H^2- FEs for $d=2$

Fact $r_{min}=5$ $(d=2)$, $\dim(P_5)=21$

$r_{min}=9$ $(d=3)$, $\dim(P_9)=220$

If. $d=1$



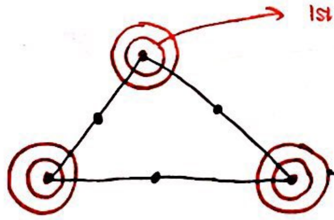
$P_r(a,c)$ $P_r(c,b)$

Fact : $r_{min}=3$ $(d=1)$. $\dim(P_3)=4$.

$\Uparrow$

What's the minimum r s.t. the piecewise polynomial function belongs to $H^2(a,c)$?

If $d=2$. $r=5$.



1st-order directional derivative (total number $=6$)
($d=2$. number $=2$)

2st-order directional derivative (total number $=9$)
$d=2$, number $=3$ ($\partial_{xy}=\partial_{yx}$)

$\dim(P_5)=21$ (DOFs)

③ Theorem. Define

"For Lagrange" element

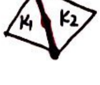
$V_r^h=\{v_h\in \boxed{C^0(\Omega)}$ or $H^1(\Omega)$; $v_h|_K\in P_r(K)\quad \forall K\in\mathcal{T}_h\}$ （V_r^h为一个抽象空间，U_r^h为V_h^r的一个刻画）

$U_r^h=\{v_h:\Omega\to R;\quad v_h|_K\in P_r(K),\ \forall K\in\mathcal{T}_h\}$
"v_h is continuous at all DOFs"

Then $V_r^h=U_r^h$.

proof : Hint. i. $V_r^h\subset U_r^h$ (trivial)

ii. $U_r^h\subset V_r^h$ (non-trivial)



e.g. P_1- element. （即证在两顶点连续，⇒ 在边上连续）

"For Hermite element" Remark : Similar result holds for H^2- FE, i.e.

let $V_r^h=\{v_h\in \boxed{C^1(\Omega)}$ or $H^2(\Omega)$; $v_h|_K\in P_r(K)$, $(r\geq 5)$. $\forall K\in\mathcal{T}_h\}$

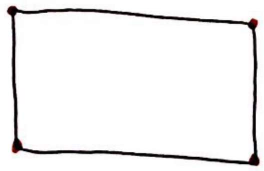
$U_r^h=\{v_h:\Omega\to R;\ v_h|_K\in P_r(K)$, $\forall K\in\mathcal{T}_h$, "$v_h$ is continuous at all DOFs".

Then $V_r^h=U_r^h$.

3. Rectangular

① Rectangular H^1- FEs.

Example (a) $d=2$, $r=1$



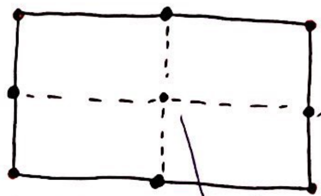
DOFs

$\dim(Q_1(K)) = (r+1)^d = 4$

[bilinear function/polynomials]

(b). $d=2$, $r=2$

DOFs



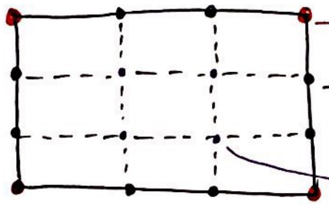
1° 顶点—必选 "保证连续性"

② "边的连续性" $r=2$ 的多项式在边上有3个自由度，只有2个顶点，补一个<u>中点</u>（位置可换，中点方便）

③ 内部选一个中点（为了方便）

$\dim(Q_2(K)) = (2+1)^2 = 9$

(c). $d=2$, $r=3$



1° 顶点连续

② 边连续（4个DOFs）补2个

$\Big\}$ 保证连续性 (C^0)

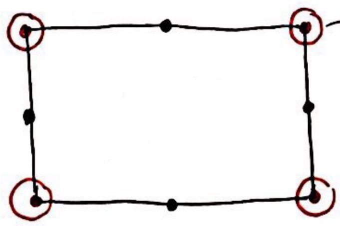
③ 内部（位置可选，方便）$\Big\}$ 保证唯一存在性.

$\dim(Q_3(K)) = 4^2 = 16$

[Bi-cubic H^1- FE Lagrange element] (when $r=3$, the Bi-cubic with Lagrange element is unique).

Remark

(d).



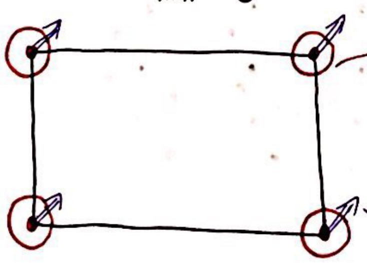
first-order derivative. ($d=2$. DOFs $+2$)

[Hermite bi-cubic H^1-FE]

When $r=3$, the choice of DOFs is "not" unique!

② H^2- FEs.

Example (a) $d=2$. $r_{min}=3$



1°. 顶点 4 + 1st-order directional derivative

2° 2nd order mixed derivative number = 4

keep "C^1".

$\dim(Q_3(K)) = 16$

[Bi-cubic H^2- FE or C^1-FE or Bogner-Fox-Schmidt element]

Fact: The above DOFs uniquely determine a bi-cubic polynomial.

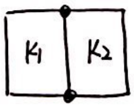
③ **Theorem** Define $V_r^h := \{ v_h \in \overset{\text{OR } H^1(\Omega)}{\boxed{C^0(\Omega)}};\ v_h|_K \in Q_r(K),\ \forall K \in \mathcal{T}_h \}$

$U_r^h := \{ v_h : \Omega \to R;\ v_h|_K \in Q_r(K),\ \forall K \in \mathcal{T}_h,\ \text{"}v_h \text{ is continuous at all DOFs"} \}$

Then $V_r^h = U_r^h$.

proof: Hint. ⓘ $V_r^h \subset U_r^h$ (trivial)

ⓘⓘ $U_r^h \subset V_r^h$ (non-trivial)

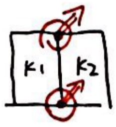


e.g. $r=1$

④ **Theorem** Define $V_r^h := \{ v_h \in \overset{\text{or } H^2(\Omega)}{\boxed{C^1(\Omega)}};\ v_h|_K \in Q_r(K),\ \forall K \in \mathcal{T}_h \}$

$U_r^h := \{ v_h : \Omega \to R;\ v_h|_K \in Q_r(K),\ \forall K \in \mathcal{T}_h,\ \text{"}v_h \text{ is continuous at all DOFs"} \}$

Then $V_r^h = U_r^h$.

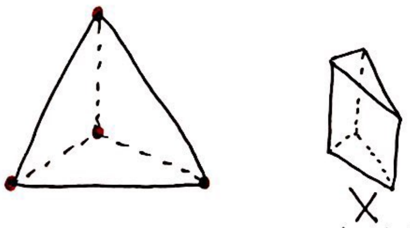


e.g. $r=3$ (BFS element)

4. 3-D Examples. (Lagrange element).

① Triangular element

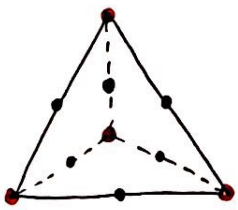
(a)



NOT Consider.

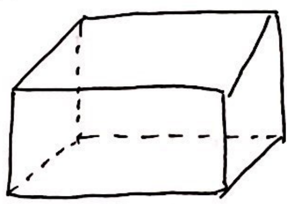
$r=1$, $\dim(P_1(K)) = \frac{(d+r)!}{d!\,r!} = \frac{(1+3)!}{1!\,3!} = 4$.

(b) $r=2$. $\dim(P_2(K)) = \frac{(2+3)!}{2!\,3!} = 10$



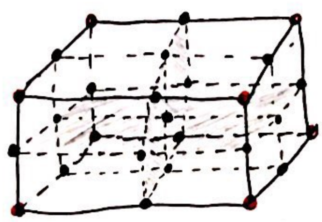
② Rectangular element

(a)



$r=1$. $\dim(Q_1(K)) = (r+1)^d = 2^3 = 8$

(b) $r=2$ $\dim(Q_2(K)) = 27$.



Finite element Method

PDE problem

$$(GP)\begin{cases}\text{Find } u\in V \text{ s.t.}\\ a(u,v)=F(v) \quad \forall v\in V\end{cases}$$

FEM

$$(GP)_h\begin{cases}\text{Find } u_h\in V_r^h \text{ s.t.}\\ a(u_h,v_h)=F(v_h) \quad \forall v_h\in V_r^h.\end{cases}$$

Remark: $(GP)_h$ is equivalent to a linear system

$$A_h\vec{\xi}_h=\vec{b}_h$$

Stiffness matrix (associated with the nodal basis).

A_h is postive definite (Assume (a)-(b) from L-M Theorem hold).

A_h is symmetric if $a(\cdot,\cdot)$ is symmetric.

Hence $(GP)_h$ has a unique solution.

Convergence

By Ceá Lemma, we have $\|u-u_h\|_V\le\frac{C}{\alpha}\inf_{v_h\in V_r^h}\|u-v_h\|_V$.

In particular, $\|u-u_h\|_V\le\frac{C}{\alpha}\|u-I_hu\|_V$, where $I_hu(x):=\sum_{j=1}^{J_h}u(\rho_j)\varphi_j(x)$.

and $\{\varphi_j\}_{j=1}^{J_h}$: (Lagrange) nodal basis.

$\{\rho_j\}_{j=1}^{J_h}$: nodes of DOFs

In general, I_hu denotes the finite interpolation of u.

Remark: Here we implicitly assume $u\in C^0(\Omega)$.

Theorem $\lim_{h\to0}\|u-u_h\|_V=0$.

proof: idea: because $\lim_{h\to0}V_r^h=V$.

To derive error estimate, we need to study the approximation properties of I_hu.

(FE interpolation theory).

(a) Assume $u\in C^{r+1}(\Omega)\cap V$ (cf. Ciarlet & Johnson)

(b) Assume $u\in W^{r+1,p}(\Omega)\cap V$ (cf. Brenner & Scott).

下午习题课

1. $a(u_h, v_h) = F(v_h)$, $\forall v_h \in V_h = \text{span}\{\varphi_1 \cdots \varphi_N\}$

Let $u_h = \sum_{j=1}^{N} u_j \varphi_j$.

then $a(\sum_{j=1}^{N} u_j \varphi_j, v_h) = F(v_h)$.

Let $v_h = \varphi_i$ ($i=1,2,\dots N$), we have

$$\begin{cases} \sum_{j=1}^{N} u_j\, a(\varphi_j, \varphi_1) = F(\varphi_1) \\ \sum_{j=1}^{N} u_j\, a(\varphi_j, \varphi_2) = F(\varphi_2) \\ \cdots \\ \sum_{j=1}^{N} u_j\, a(\varphi_j, \varphi_N) = F(\varphi_N) \end{cases} \Longrightarrow \underset{\substack{\uparrow \\ A:\ a_{ij} = a(\varphi_j, \varphi_i)}}{[A]} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} F(\varphi_1) \\ \vdots \\ F(\varphi_N) \end{bmatrix}$$

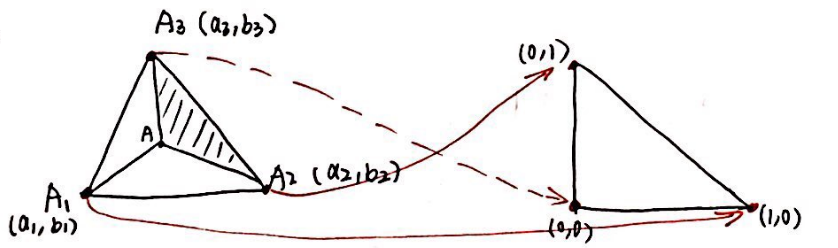
Example:

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \Longrightarrow a(u,v) = \int \nabla u \cdot \nabla v \, dx$$

then

$$A = \begin{bmatrix} \cdot & \cdot & \\ \vdots & \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \\ \vdots & \sum_{K\in\mathcal{T}_h} \int_K \nabla\varphi_j, \nabla\varphi_i \, dx \\ \vdots & \end{bmatrix} = \sum_{K\in\mathcal{T}_h} \begin{bmatrix} \int_K \nabla\varphi_1 \nabla\varphi_1 dx & \cdots & \int_K \nabla\varphi_N \nabla\varphi_1 dx \\ & & \\ \int_K \nabla\varphi_1 \nabla\varphi_N dx & \cdots & \int_K \nabla\varphi_N \nabla\varphi_N dx \end{bmatrix}$$

2.



映射到标准三角形

$(x_1,x_2) \to (\lambda_1, \lambda_2)$ $\quad \int_K \cdots dx_1 dx_2 = \int_{\hat K} \cdots (2S)\, d\lambda_1 \lambda_2$,

$A_1 \to (1,0)$ where

$A_2 \to (0,1)$ $\quad S = \Delta A_1A_2A_3$

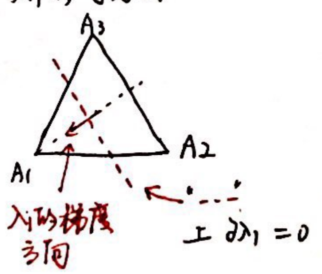
$A_3 \to (0,0)$ $\quad = \begin{vmatrix} a_1-a_3, & a_2-a_3 \\ b_1-b_3 & b_2-b_3 \end{vmatrix}$

$$\begin{cases} \lambda_1 = \dfrac{S_{\Delta AA_2A_3}}{S_{\Delta A_1A_2A_3}} \\ \lambda_2 = \dfrac{S_{\Delta A_1AA_3}}{S_{\Delta A_1A_2A_3}} \\ \lambda_3 = \dfrac{S_{\Delta A_1A_2A}}{S_{\Delta A_1A_2A_3}} \end{cases} \quad \text{then} \quad \begin{cases} \lambda_1 = \begin{vmatrix} x_1 & x_2 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \\ \lambda_2 = \begin{vmatrix} a_1 & b_1 & 1 \\ x_1 & x_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \\ \lambda_3 = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ x_1 & x_2 & 1 \end{vmatrix} \end{cases}$$

then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A_1 - A_3, A_2 - A_3) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + A_3$

$$= \underbrace{\begin{bmatrix} a_1-a_3, & a_2-a_3 \\ b_1-b_3, & b_2-b_3 \end{bmatrix}}_{\substack{\uparrow \\ S = \begin{vmatrix} a_1-a_3, & a_2-a_3 \\ b_1-b_3, & b_2-b_3 \end{vmatrix}}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$$

Remark: λ_1的等高线



3.

① $L^p(\Omega)$

② Minkowski inequality $\quad 1\le p\le\infty,\ f,g\in L^p(\Omega)$, then

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

Hölder inequality $\quad 1\le p,q\le\infty,\ 1=\frac{1}{p}+\frac{1}{q},\ f\in L^p(\Omega),\ g\in L^q(\Omega)$, then $f\cdot g\in L^1(\Omega)$, and

③ Let $L^1_{loc}(\Omega)=\{f: f\in L^1(D),\ \forall$ compact set $D\subset\Omega\}$

If $f\in L^1_{loc}(\Omega)$, then f is equivalent to a distribution

$$\langle f,\varphi\rangle=\int_\Omega f(x)\varphi(x)\,dx \qquad \forall\varphi\in\mathcal{D}(\Omega).$$

称 $f\in L^1_{loc}(\Omega)$ 具有广义导数 $D^\alpha_w f$，若 $\exists\, g\in L^1_{loc}(\Omega)$, s.t.

$$\int_\Omega g(x)\cdot\varphi(x)\,dx=(-1)^{|\alpha|}\int_\Omega f(x)\cdot\partial^\alpha\varphi(x)\,dx,\quad \forall\varphi\in\mathcal{D}(\Omega).$$

并记为 $D^\alpha_w f=g$.

④ Sobolev space.

f 广义导数 $(D^\alpha f)$，经典导数 $(\partial^\alpha f)$.

Let $m\geq 0$ be an integer, then.

Norm
$$\begin{cases}\|v\|_{W^{m,p}(\Omega)}=\|v\|_{m,p,\Omega}=\left\{\sum_{|\alpha|\le m}\|D^\alpha v\|^p_{0,p,\Omega}\right\}^{\frac{1}{p}}, & 1\le p<\infty.\\ \|v\|_{W^{m,\infty}(\Omega)}=\|v\|_{m,\infty,\Omega}=\max_{|\alpha|\le m}\|D^\alpha v\|_{0,\infty,\Omega}\end{cases}$$

then Sobolev Space is defined by

$$W^{m,p}(\Omega)=\{v\in L^1_{loc}(\Omega): \|v\|_{m,p,\Omega}<\infty\}$$

$$W^{m,2}(\Omega)=H^m(\Omega),\ \text{with}\ (u,v)_{m,\Omega}=\sum_{|\alpha|\le m}(D^\alpha u, D^\alpha v)$$

Banach space $\to$ Hilbert space

⑤ 嵌入：$X\hookrightarrow Y$（X嵌入到Y）if (i) $X\subset Y$

(ii) X到Y有连续内射; i.e. $\exists\, C=const>0$, $\|x\|_Y\le C\|x\|_X$, $\forall x\in X$

紧嵌入：$X\Subset Y$（X紧嵌入到Y）if (i) $X\hookrightarrow Y$

(ii) X到Y的内射为紧的（X中有界集为Y的紧集）

i.e. $X\Subset Y$

$\{x_n\}\subset X$ & $\|x_n\|_X\le M$, then $\exists\, n_k$, s.t. x_{n_k}依Y-norm收敛

Hölder continuity:

If $v\in C^{0,\alpha}(\bar\Omega)$, then $\exists\, M>0$ s.t. $|v(x)-v(y)|\le M\|x-y\|^\alpha$, $\forall x,y\in\bar\Omega$, where $\alpha\in(0,1]$.

$C^{m,\alpha}(\bar\Omega)$ ($m\in\mathbb{N}$, $\alpha\in(0,1]$): $C^m(\bar\Omega)$中m阶导数为α次Hölder连续的函数全体

Banach space. where
$$\|v\|_{C^{m,\alpha}}=\|v\|_{m,\infty,\Omega}+\max_{|\beta|=m}\sup_{\substack{x,y\in\bar\Omega\\ x\ne y}}\frac{|\partial^\beta v(x)-\partial^\beta v(y)|}{\|x-y\|^\alpha}.$$

⑥ [嵌入定理] 设 $m\geq 0$ 为整数, $1\leq p<\infty$, 且设 Ω 为 $\mathbb{R}^n$ 中区域, 即有界开的连通集合, 且有 Lipschitz 连续边界, 则

$$W^{m,p}(\Omega)\hookrightarrow\begin{cases} L^{p^*}(\Omega),\quad \frac{1}{p^*}=\frac{1}{p}-\frac{m}{n},\quad \text{if } m<\frac{n}{p} & [m-\frac{n}{p}<0] \\ L^{q}(\Omega),\quad \forall q\in[1,\underset{\substack{\uparrow \\ \text{NOT include } \infty}}{\underline{\underline{\infty)}}},\ \text{if } m=\frac{n}{p}\ ;\ \text{If } p=1,\ q & [m-\frac{n}{p}=0] \\ C^{0,m-\frac{n}{p}}(\bar\Omega), \qquad \text{if } \frac{n}{p}<m<\frac{n}{p}+1 & [m-\frac{n}{p}\in(0,1)] \\ C^{0,\alpha}(\bar\Omega),\ \forall 0<\alpha\underset{\text{NOT include } 1}{\underline{\underline{<}}}1, \qquad \text{if } m=\frac{n}{p}+1 & [m-\frac{n}{p}=1] \\ C^{0,1}(\bar\Omega), \qquad \text{if } \frac{n}{p}+1<m & [m-\frac{n}{p}>1] \end{cases}$$

Remark 2. If $m-\frac{n}{p}=5.5$. then $\begin{cases} m-4-\frac{n}{p}=1.5 \text{ then } W^{m-4,p}(\Omega)\hookrightarrow C^{0,1}(\bar\Omega), \text{ Hence } W^{m,p}\hookrightarrow C^{4,1}(\bar\Omega) \\ m-5-\frac{n}{p}=0.5 \text{ then } W^{m-5,p}(\Omega)\hookrightarrow C^{0,0.5}(\Omega), \text{ hence } W^{m,p}\hookrightarrow C^{5,0.5}(\bar\Omega). \end{cases}$

⑦ [紧嵌入定理]

$$W^{m,p}(\Omega)\Subset\begin{cases} L^{q}(\Omega),\ \forall 1\leq q<p^*,\ \frac{1}{p^*}=\frac{1}{p}-\frac{m}{n},\quad \text{if } m<\frac{n}{p} & (1) \\ L^{q}(\Omega),\ \forall q\in[1,\infty) \quad ,\ \text{if } m=\frac{n}{p} & (2) \\ C^{0}(\bar\Omega) \quad ,\ \text{if } m>\frac{n}{p} & (3) \end{cases}$$

Remark: (i). $H^1(\Omega)\Subset L^2(\Omega)$, for any n. (Rellich Theorem)

when $n=1$ $m-\frac{n}{p}=1-\frac{1}{2}=\frac{1}{2}$. By (3) we have $H^1(\Omega)=W^{1,2}(\Omega)\Subset C^0(\bar\Omega)$.

$C^0(\bar\Omega)\subset L^3(\Omega)$ and $L^3(\Omega)\Subset L^2(\Omega)$,

then $H^1(\Omega)\Subset L^2(\Omega)$

when $n=2$, $m-\frac{n}{p}=1-\frac{2}{2}=0$. By (2), $H^1(\Omega)=W^{1,2}(\Omega)\Subset L^q(\Omega)$, $\forall q\in[1,\infty)$

Choosing $q=2$. then $H^1(\Omega)\Subset L^2(\Omega)$.

when $n>2$. $m-\frac{n}{p}<0$, By (3) $H^1(\Omega)\Subset L^q(\Omega)$, $\forall 1\leq q<p^*$, where $p^*=\frac{1}{\frac{1}{p}-\frac{m}{n}}>p$

By Choosing $q=p$, then $H^1(\Omega)\Subset L^2(\Omega)$.

where we use $\underline{L^q(\Omega)\Subset L^p(\Omega)}$ if $p<q$.

Since if $m=0$, we have $W^{0,p}(\Omega)\Subset L^q(\Omega)$, $\forall 1\leq q<p$ (By (1)). $\quad W^{0,p}(\Omega)=L^p(\Omega)$

(ii) $W^{1,p}(\Omega)\Subset L^p(\Omega)$, $W^{k+1,p}(\Omega)\Subset W^{k,p}(\Omega)$, for every n.

- since for $W^{1,p}(\Omega)$, $m=1$, then $p^*>p$ for any n.
  Since $1=m<\frac{n}{p}=n$ always holds, By Choosing $q=p<p^*$ in (1) we have $W^{1,p}(\Omega)\Subset L^p(\Omega)$ for every n.
- 略.

(iii) If $n=2$. then $H^2(\Omega)\hookrightarrow C^0(\bar\Omega)$, and $H^2(\Omega)\Subset C^0(\bar\Omega)$, but $H^1(\Omega)\not\hookrightarrow C^0(\bar\Omega)$.

- If $n=2$. $H^2(\Omega)=W^{2,2}(\Omega)$, then $m-\frac{n}{p}=2-\frac{2}{2}=1$ By (3) & 嵌入定理
  we have $\begin{cases} H^2(\Omega)\hookrightarrow C^{0,\alpha}(\bar\Omega). \text{ since } \begin{cases} C^{0,\alpha}(\bar\Omega)\subset C^0(\bar\Omega) \\ \|\cdot\|_{C(\bar\Omega)}\leq\|\cdot\|_{C^{0,\alpha}(\bar\Omega)} \end{cases} \text{ then } H^2(\Omega)\hookrightarrow C^0(\bar\Omega) \\ H^2(\Omega)\Subset C^0(\bar\Omega). \end{cases}$
- If $n=2$, $H^1=W^{1,2}$ then $m-\frac{n}{p}=1-\frac{2}{2}=0$, No (3) !

⑧ Theorem (Deny - Lions) [范数等价定理]

Let Ω be a bounded Lipschitz domain, for any $k \geq 0$, $\exists$ constant $C(\Omega)$ such that

$$\inf_{p \in P_k(\Omega)} \|v+p\|_{H^{k+1}(\Omega)} \leq C(\Omega) |v|_{H^{k+1}(\Omega)} \quad \forall v \in H^{k+1}(\Omega) \tag{8.1}$$

Here $P_k(\Omega)$ is the set of polynomails over Ω with degree $\leq k$.

proof : Let $N = \dim P_k(\Omega)$ and f_i, $1 \leq i \leq N$, be a basis of the dual space of $P_k(\Omega)$.

idea : Let $\underset{\in P_k(\Omega)}{f_i} \xrightarrow{\text{Hahn-Banach extension}} \underset{\in H^{k+1}(\Omega)}{f_i}$, $1 \leq i \leq N$ s.t. $f_i(p) = 0$ $1 \leq i \leq N$ iff $p = 0$ for any $p \in P_k(\Omega)$.

If we can show that $\exists$ constant $C(\Omega)$ such that

$$\|v\|_{H^{k+1}(\Omega)} \leq C(\Omega) \left(|v|_{H^{k+1}(\Omega)} + \sum_{i=1}^{N} |f_i(v)| \right) \quad \forall v \in H^{k+1}(\Omega). \tag{8.2}$$

Then (8.1) is a consquence of (8.2) because for any $v \in H^{k+1}(\Omega)$, there exists a $p \in P_k(\Omega)$ such that $f_i(p) = - f_i(v)$, $1 \leq i \leq N$.

Lemma (Bramble - Hilbert) Let Ω be a bounded Lipschitz domain and let $k \geq 0$ be an integer. Denote by $X = H^{k+1}(\Omega)$. Let Y be a Banach space and let $f \in \mathcal{L}(X, Y)$ be a continuous linear operator from X to Y such that $f(p) = 0$ for any $p \in P_k(\Omega)$. Then there exists a constant $C(\Omega)$ such that

$$\|f(v)\|_Y \leq C(\Omega) \|f\|_{\mathcal{L}(X,Y)} |v|_{H^{k+1}(\Omega)} \quad \forall v \in H^{k+1}(\Omega). \tag{8.3}$$

where $\|\cdot\|_{\mathcal{L}(X,Y)}$ is the operator norm.

proof : Idea : since $f(p) = 0$ for any $p \in P_k(\Omega)$.

then $\|f(v)\|_Y = \|f(v+p)\|_Y \leq \|f\|_{\mathcal{L}(X,Y)} \|v+p\|_X = \|f\|_{\mathcal{L}(X,Y)} \|v+p\|_{H^{k+1}(\Omega)}$

then $\|f(v)\|_Y = \inf_{p \in P_k(\Omega)} \|f(v+p)\|_Y \leq \|f\|_{\mathcal{L}(X,Y)} \inf_{p \in P_k(\Omega)} \|v+p\|_{H^{k+1}(\Omega)}$

By $\inf_{p \in P_k(\Omega)} \|v+p\|_{H^{k+1}(\Omega)} \leq C(\Omega) |v|_{H^{k+1}(\Omega)}$ in [Theorem Deny - Lions]

we have

$$\|f(v)\|_Y \leq \|f\|_{\mathcal{L}(X,Y)} C(\Omega) |v|_{H^{k+1}(\Omega)}.$$

Class 4.

III Finite Element Method (as a special Galerkin method). $(V_N = V_h)$

1. Two main ingredient: mesh & shape function space (polynomials)
2. Nodal basis
3. Examples of FE spaces

① H^1-FES (C^0-FES)

(a) Triangular element: P_1, P_2, P_3-FE space "Language vs Hermite"

(b) Rectangular element: Q_1, Q_2, Q_3-FE space.

② H^2-FEs (C^1-FEs)

(a) Triangular element: Argyris & Bell elements.

(b) Rectangular element: Bogner-Fox-Schmidt (BFS) schemes

③ Characterization theorem $\boxed{V_r^h = U_r^k}$

4. FE interpolation theory & error estimates

① two cases: $C^k(\bar{\Omega})$ & $W^{k,p}(\Omega)$

② Céa Lemma gives

$$\|u-u_h\|_{V,\Omega} \le \frac{C}{\alpha}\|u - \underbrace{I_h u}_{\text{(FE interpolation operator)}}\|_{V,\Omega}$$

$$= \sum_{K\in T_h} \frac{C}{\alpha}\|(u-I_h u)|_K\|_{V,K} = \frac{C}{\alpha}\sum_{K\in T_h}\| u|_K - \underline{(I_h u)|_K}\|_{V,K}$$

$$= \frac{C}{\alpha}\sum_{K\in T_h}\|u|_K - \underbrace{I_{h,K}u}_{\in V_r(K)}\|_{V,K}$$

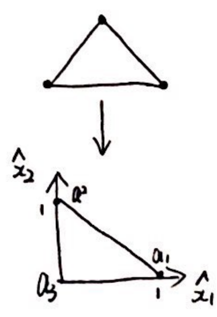
将分片函数插值转化成单元K上的函数插值。因在K上为多项式，从而将分片函数插值转化为多项式插值.

- e.g. $V = H_0^1(\Omega) \Rightarrow \begin{cases} \|v\|^2_{V,\Omega} = \int_\Omega |\nabla v|^2 dx + \int_\Omega |v|^2 dx \\ \|v\|^2_{V,K} = \int_K |\nabla v|^2 dx + \int_K |v|^2 dx \end{cases}$

- claim: $(I_h u)|_K = \underline{I_{h,K} u}$

local interpolation of u into the underlying shape function space $V_r(K)$

e.g. For P_1-FE method shape function space is $P_1(K)$

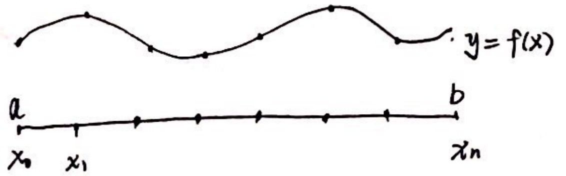


$\hat{\varphi}_1(x) = \hat{x}_1$

$\hat{\varphi}_2(x) = \hat{x}_2$

$\hat{\varphi}_3(x) = 1 - \hat{x}_1 - \hat{x}_2$.

- $n=1$: poly插值.



Find $P_n \in P_n([a,b])$, s.t.

$$P_n(x_j) = f(x_j) \quad j = 0,1,2,\dots n.$$

If $\{x_j\}_{j=0}^n$ are distinct, then $\exists P_n$, and $P_n(x) = \sum_{j=0}^n f(x_j) L_{n,j}(x)$, where $\{L_{n,j}(x)\}_{j=0}^n$ are Lagrange polys.

$\{L_{n,j}\}_{j=0}^n$ forms a basis for $P_n([a,b])$, i.e., $P_n([a,b]) = \text{span}\{L_{n,j}(x),\ j=0,1,\dots n\} = \text{span}\{1, x, x^2, \dots x^n\}$

Exercise 1. Verify this result and

$$f(x) - P_n(x) = \frac{f^{n+1}(\xi_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

where $\min\{x, x_j\} \le \xi_x \le \max\{x, x_j\}$

Example: Consider $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$

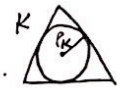
and $\boxed{V = H_0^1(\Omega)}$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, V_h: P_1 - FE space.

$$F(v) = \int_\Omega f v \, dx.$$

then

(a) $\|u - I_{h,K} u\|_{V,K} = \left[\int_K |\nabla(u - I_{h,K} u)|^2 dx\right]^{\frac{1}{2}}$

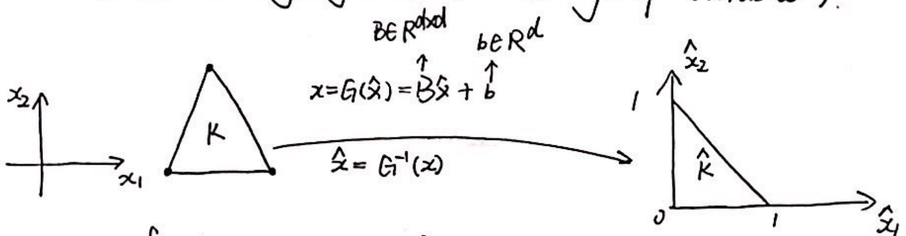
$$\le \frac{C h_K^2}{\rho_K} |u|_{H^2}, \quad \text{where} \begin{cases} h_K = \operatorname{diam}(K) = \max\limits_{x,x' \in K} |x - x'|. \\ \rho_K \text{ is the radius of the largest incircle.} \end{cases}$$



$\rho_K \le h_K$ $\quad (0 \le \frac{h_K}{\rho_K} \le \delta)$.

(b). $\|u - I_{h,K} u\|_{L^2(K)}^2 \le C h_K^2 |u|_{H^2}$.

proof: idea: Scaling argument (i.e. change of variables).



$$\int_K | u - I_{h,K} u |^2 dx$$

$$= \int_{\hat{K}} | \underline{\hat{u} - \hat{I}_{h,K} \hat{u}} |^2 |\det(B)| \, d\hat{x} \qquad \Rightarrow \|u - I_{h,K} u\|_{L^2(K)} \le \frac{C}{(\rho_{\hat{K}})^2} h_K^2 |u|_{H^2}.$$

Jacobi

Deny - Lions Thm

$\|\hat{u} - \hat{I}_{h,K}\hat{u}\|_{L^2(\Omega)}^2 \le \inf\limits_{p \in P_1(\hat{K})} \|\hat{u} + p\|_{H^2(\hat{K})}^2 \le C(\Omega) |\hat{u}|_{H^2(\hat{K})}^2$

$\hat{I}_{h,K}\hat{u} \in P_1$ (V_h: P_1 - FE)

$\le C |\det(B)| \, \underline{|\hat{u}|_{H^2(\hat{K})}^2}$

By (ii) $m=2$ $\le C \|B\|^4 |u|_{H^2(K)}^2$

By (iii) $\le C \left(\frac{h_K}{\rho_{\hat{K}}}\right)^4 |u|_{H^2(K)}^2$, where $\rho_{\hat{K}}$ is a constant.

Fact:

(i) $\det(B) \ne 0 \iff \operatorname{meas}(K) \ne 0$

(ii) $|\hat{u}|_{H^m(\hat{K})} \le C \|B\|^m |\det(B)|^{-\frac{1}{2}} |u|_{H^m(K)}$

(iii) $|u|_{H^m(K)} \le C \|B^{-1}\|^m |\det(B)|^{\frac{1}{2}} |\hat{u}|_{H^m(\hat{K})}$

(iv) $\|B\| \le \frac{h_K}{\rho_{\hat{K}}}$ constant $\qquad \|B\| = O(h^K)$

$\|B^{-1}\| \le \frac{h_{\hat{K}}}{\rho_K} \le h_{\hat{K}} \delta h_K^{-1}$. $\quad \left(\text{where } \frac{1}{\rho_K} = \frac{h_K}{\rho_K} \cdot \frac{1}{h_K} \le \delta h_K^{-1}\right) \qquad \|B^{-1}\| = O(h_K^{-1})$

operator norm

(vi) $\left(\frac{\rho_K}{\hat{h}_K}\right)^d \le |\det B| \le \left(\frac{h_K}{\rho_{\hat{K}}}\right)^d \qquad |\det(B)| = O(h_K^d)$

Similar results holds for high order element.

Theorem Suppose $\boxed{u \in H^s(\Omega)} = W^{s,2}(\Omega)$ $(s>1)$ and $\mathcal{T}_h$ is shape-regular mesh, then

(a) $\|u - I_{h,K} u\|_{L^2(K)} \le C h_K^{\mu} |u|_{H^{\mu}(K)}$

(b) $|u - I_{h,K} u|_{H^1(K)} \le C h_K^{\mu-1} |u|_{H^{\mu}(K)}$

$\mu = \min\{r+1, s\}$

Hence,

(a)' $$\|u - I_h u\|_{L^2(\Omega)} \le C \sum_{K\in\mathcal{T}_h} h_K^{\mu} |u|_{H^{\mu}(K)} \le C h^{\mu} |u|_{H^{\mu}(\Omega)}.$$

(b)' $$|u - I_h u|_{H^1(\Omega)} = \|\nabla(u - I_h u)\|_{L^2(\Omega)} \le C \sum_{K\in\mathcal{T}_h} h_K^{\mu-1} |u|_{H^{\mu}(K)} \le C h^{\mu-1} |u|_{H^{\mu}(\Omega)}$$

where $h := \max_{K\in\mathcal{T}_h} \{h_K\}$.

Remark : The order of error estimates depend on poly degree r and regularity of the underlying function. (indicated by s).

By Ceá Lemma and interpolation error estimates, we get

Theorem : Suppose exact solution of (GP) $u \in H^s(\Omega)$ $(s>1)$ let u_h be solution of P_r - FE method, then

"Energy" Norm error estimate.

$$\|u-u_h\|_V = \|u - u_h\|_{H^1(\Omega)} \le C h^{\mu-1} |u|_{H^{\mu}(\Omega)} \qquad \circledast\text{-}1$$

where $\|u-u_h\|_{H^1(\Omega)} = \sqrt{\|u-u_h\|^2_{L^2(\Omega)} + \|\nabla(u-u_h)\|^2_{L^2}}$, $\mu = \min\{r+1, s\}$.

$$(GP)\quad \begin{cases} -\Delta u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Q. How about error estimate in norms other than energy norm ?

(a) L^2 - norm estimate.

Theorem : Suppose $u \in H^s(\Omega)$ $(s>1)$ and $(GP)^*$ is "regular enough", i.e. suppose $\exists!\ \varphi \in H^2(\Omega)$ and $\|\varphi\|_{H^2(\Omega)} \le C\|e_h\|_{L^2(\Omega)}$

2nd order elliptic. since $L^*\varphi = e_h$.

then $\|u-u_h\|_{L^2(\Omega)} \le Ch^{\mu}|u|_{H^{\mu}(\Omega)}$ ⊗-2

where $\mu = \min\{r+1, s\}$.

proof : Idea : Duality (Aubin - Nitsche) argument.

Consider the dual problem of (GP), find $\varphi \in H^2(\Omega)$, s.t.

$$\left.\begin{array}{l} a(v,\varphi) = (e_h, v)_{L^2(\Omega)}, \quad \forall v \in V \\ \text{where } e_h := u - u_h. \end{array}\right\} (GP)^*$$

Since $e_h \in V$, then Let

$$\|u-u_h\|^2_{L^2(\Omega)} = \|e_h\|^2_{L^2(\Omega)} = (e_h, \underbrace{e_h}_{\in V})_{L^2(\Omega)} = a(e_h, \varphi)$$

$$= \underbrace{a(e_h,\varphi) - a(e_h, \varphi_h)}_{\text{By Galerkin orthogonality. this term} = 0}, \quad \varphi_h = I_h\varphi \in V_h$$

$$= a(e_h, \varphi - I_h\varphi)$$

$$\le C\|e_h\|_V \|\varphi - I_h\varphi\|_V$$

Here $V = H^1_0(\Omega)$ OR $V = H^1(\Omega)$

⊗-1 ; (b)' in Thm (P_{26}) $(\mu = 2)$

$$\le Ch^{\mu-1}|u|_{H^{\mu}(\Omega)} \quad Ch|\varphi|_{H^2(\Omega)} \le Ch\|e_h\|_{L^2(\Omega)}$$

Assumption. $|\varphi|_{H^2(\Omega)} \le C\|e_h\|_{L^2(\Omega)}$

$$\le Ch^{\mu}|u|_{H^{\mu}(\Omega)} \cdot \|u-u_h\|_{L^2}$$

Hence $\|u-u_h\|_{L^2(\Omega)} \le Ch^{\mu}|u|_{H^{\mu}(\Omega)}$

Remark : ⊗-1 and ⊗-2 are "optimal" error estimates.

with respect to interpolation errors.

(b) L^{∞} - norm error estimate.

Theorem : For Possion problem, there hold

(i) For $r \ge 1$ $\|\nabla u - \nabla u_h\|_{L^{\infty}(\Omega)} \le Ch|u|_{W^{2,\infty}(\Omega)}$

(ii) If $r=1$

$$\|u-u_h\|_{L^{\infty}(\Omega)} \le Ch^2 \underbrace{|\ln h|}_{h^{2-\varepsilon}\ (\varepsilon>0)} |u|_{W^{2,\infty}(\Omega)}$$

(iii) If $r>1$

$$\|u-u_h\|_{L^{\infty}(\Omega)} \le Ch^2|u|_{W^{2,\infty}(\Omega)}$$

proof : Idea : Weighted Sobolev space Technique.
(Nitsche's technique)

Remark : "Similar" interpolation & error estimates hold for H^2- FE (or H^1-FE) methods.

Energy Space / Norm for 4^{th} order problem is

$V = H^2(\Omega)$,

(i) $\|u-u_h\|_{H^2(\Omega)} \le Ch^{\mu-2}|u|_{H^\mu(\Omega)}$.

where $\mu = \min\{r+1, s\}$, $s>1$, and

(ii) $\|u-u_h\|_{H^j(\Omega)} \le Ch^{\mu-j}|u|_{H^\mu(\Omega)}$, $j=0,1$.

which can be derived by duality argument.

A priori & posteriori error estimates

(a) A prior error estimates.

$\|u-u_h\|_* \le Ch^\mu |u|_{H^\mu(\Omega)} = \underbrace{\hat{C}h^\mu}_{\text{NOT computable}}$, where $\hat{C} = C|u|_{H^\mu(\Omega)}$, $\hat{C}$ depends on the exact solution u (which is not known).

(b) A posteriori error estimates.

$\|u-u_h\|_+ \le \underbrace{C(u_h)h^\nu}_{\text{Computable!}}$, where $C(u_h)$ is a positive constant depend on the FE solution u_h (which is known)

Remark : (a) implies convergence

(b) predict the quality of u_h & can be used for design adaptive grid method.

Solvers (solution methods)

Q : How to solve $\underbrace{A_h}_{\text{Stiffness metrix.}}\vec{\xi} = \vec{b}$?

Facts : (i) A_h is sparse. it's good for iterative methods. (CG, Krylor space method)

(ii) $\mathrm{Cond}_2(A_h) = \|A_h\|_2 \|A_h^{-1}\|_2 = O(h^{-m})$ (m : order of PDE. e.g. m=2,4)

Remark : A pre-conditioning technique is often used to solve $A_h\vec{\xi} = \vec{b}$.

预处理器 $\longrightarrow$ $B_h A_h \vec{\xi} = B_h \vec{b}$

s.t. $\mathrm{cond}_2(B_h A_h) << \mathrm{cond}_2(A_h)$

5. Extensions

① Extension of Galerkin Framework.

② $V_N \not\subset V$ or $V_h \not\subset V$ (Variational Crimes).

Remark:

$$(GP)\begin{cases} \text{Find } u \in V \text{ s.t.} \\ a(u,v) = F(v) \quad \forall v \in V \end{cases}$$

Example (a)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

integration by parts twice.

$$\xRightarrow{\text{Integration By parts}} \int_\Omega -\Delta u \cdot v \, dx$$

$$= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \cancel{\frac{\partial u}{\partial n}} \underset{0}{\overset{\shortparallel}{v}} \, ds = \int_\Omega f v \, dx$$

$$\xRightarrow{\text{Integration By parts.}} -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} \underset{0}{\overset{\shortparallel}{u}} \cancel{\frac{\partial v}{\partial n}} \, ds = \int_\Omega f v \, dx$$

$$\Longrightarrow \underbrace{-\int_\Omega \overset{u \in L^2}{u} \, \Delta \overset{v \in H^2}{v} \, dx}_{\tilde{a}(u,v)} = \underbrace{\int_\Omega f v \, dx}_{F(v)} \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega)$$

$$(\widetilde{GP})\begin{cases} \text{Find } u \in \underbrace{L^2(\Omega)}_{\text{Trial space } U} \text{ s.t.} \\ \tilde{a}(u,v) = F(v), \quad \forall v \in \underbrace{V = H^2(\Omega) \cap H_0^1(\Omega)}_{\text{Test space } V} \end{cases}$$

$$\left[\begin{matrix} U = L^2(\Omega) \\ V = H^2(\Omega) \cap H_0^1(\Omega) \end{matrix} \quad V \neq U \right]$$

Example (b) Non-divergence form (No Integration by parts).

$$A(x): D^2 u = f \quad \text{in } \Omega \qquad \text{where } D^2 u : \text{Hessian of } u.$$

$$u = 0 \quad \text{on } \partial\Omega \qquad \mathrm{tr}(D^2 u) = \Delta u$$

Assume $A \in [C^0(\bar{\Omega})]^{d\times d}$ s.t.

$$\lambda |\vec{\xi}|^2 \le A\vec{\xi} \cdot \vec{\xi} \le \Lambda |\vec{\xi}|^2 \quad \forall \vec{\xi} \in \mathbb{R}^d$$

$$(0 < \lambda < \Lambda < \infty).$$

$$\text{Define } \begin{cases} a(u,v) = \int_\Omega A(x) : D^2 u(x) \cdot v(x) \, dx. \\ F(v) = \int_\Omega f v \, dx \end{cases}$$

$$\begin{cases} \text{Find } u \in \underbrace{H^2(\Omega) \cap H_0^1(\Omega)}_{\text{Trial space } U} \text{ s.t.} \\ a(u,v) = F(v) \quad \forall v \in \underbrace{L^2(\Omega)}_{\text{Test space } V} \end{cases} \qquad [U \neq V]$$

下午习题课

1、 迭代误差衰减速度：

$$\|u^m - u\|_V \le 2\left(\frac{\sqrt{\mathrm{cond}(A)}-1}{\sqrt{\mathrm{cond}(A)}+1}\right)^m \|u^0 - u\|_V$$ where A is the metrix in $Ax=b$.

因而 If cond(A) 过大，迭代速度很慢 $\left(\frac{\sqrt{\mathrm{cond}(A)}-1}{\sqrt{\mathrm{cond}(A)}+1} \approx 1\right)$.

Recall that $\mathrm{cond}(A) = O(h^{-2})$.

consider $d=1$

$$\begin{cases} -u'' = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

the operator $L = -\frac{d^2}{dx^2}$. let $Lu = \lambda u$

then $-u'' - \lambda u = 0$

$\Rightarrow u'' + \lambda u = 0$

d: dimension

then $\begin{cases} u_n = \sin nx \text{ . eigenfunction} \\ \lambda_n = n^2 \quad \text{eigenvalue.} \end{cases}$

$[\lambda_n = O(n^{\frac{2}{d}})]$

$\lambda_n = O(n^2)$

Recall metrix $\tilde{A} = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & & 2 \end{bmatrix}_{N\times N}$, the eigenvalues of this $\tilde{A}$ approximate n^2. $(n=1,2,\dots N)$

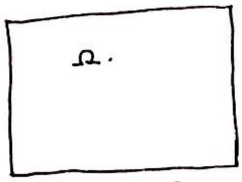
since $\lambda_n = O(n^{\frac{2}{d}})$. If $d=1$.

$$\mathrm{cond}(A) = \underset{(n^2)}{\max \lambda_n} = O(n^2) = O(h^{-2})$$

$\uparrow$ $n = 1/h$.

Hence $\mathrm{cond}(A) = O(h^{-2})$.

$d=2$



$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

operator $L = -\Delta$

the corresponding eigen-pair is

$\begin{cases} u_{n,m} = \sin nx \cdot \sin my & \text{eigenfunction} \\ \lambda_{n,m} = m^2 + n^2 & \text{eigenvalue.} \end{cases}$

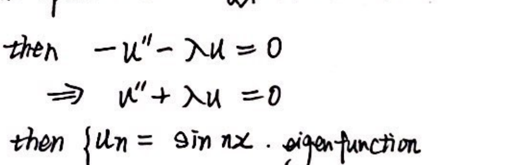
$\lambda_n = O(n)$

If $d=2$

$$\mathrm{cond}(A) = \underset{(2n^2)}{\max \lambda_n} = O(n^2) = O(h^{-2})$$

If $d=3$ $\max \lambda_n = 3n^2$.

2 标准△转换下求 Matrix.



$$\begin{cases} \varphi_1 = 2(\lambda_1+\lambda_2-1)(\lambda_1+\lambda_2-\frac{1}{2}) \\ \varphi_2 = 2\lambda_1(\lambda_1-\frac{1}{2}) \\ \varphi_3 = -4\lambda_1(\lambda_1+\lambda_2-1). \end{cases}$$

In stiffness Matrix, we need to compute

$$\int_K \nabla\phi_i \cdot \nabla\phi_j \, dx$$

$$\int_K \nabla\phi_i \cdot \nabla\phi_j \, dx = \int_K \hat{\nabla}\phi_i(\lambda_1,\lambda_2) \cdot \hat{\nabla}\phi_j(\lambda_1,\lambda_2) \cdot 2|K| \, d\lambda_1 d\lambda_2$$

If Mass Matrix, we need to compute.

$$\int_K \phi_i \cdot \phi_j \, dx$$

$$\int_K \phi_i \phi_j \, dx = \int_K \phi_i(\lambda_1(x_1,x_2), \lambda_2(x_1,x_2)) \cdot \phi_j(\lambda_1(x_1,x_2), \lambda_2(x_1,x_2)) dx$$
$$= \int_K \phi_i(\lambda_1,\lambda_2) \cdot \phi_j(\lambda_1,\lambda_2) \cdot 2|K| \, d\lambda_1 d\lambda_2$$

Fact: $\int_K \lambda_1^p \lambda_2^q \lambda_3^r \, dx = \frac{p!q!r!}{(p+q+r+2)!} \cdot 2|K|$, $p, q, r \geq 0$ (integer)

proof: Idea: $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = B($

3. Duality Problem.

Consider dual operator $(Lu, v) = (u, L^* v)$

since $(-\Delta u, v) = (u, -\Delta v)$, hence $-\Delta u$ is self-dual operator.

If $Lu = f$ (原方程)

$\Rightarrow (Lu, v) = (f, v)$

$\Rightarrow \underline{a(u,v)} = (f, v)$

If $L^* u = f$ (共轭方程)

$\Rightarrow (L^* u, v) = (f, v)$

$\Rightarrow (u, Lv) = (f, v)$

$\Rightarrow (Lv, u) = (f, v)$

$\Rightarrow \underline{a(v,u)} = (f, v)$

L的自共轭性反映 $a(\cdot,\cdot)$ 的对称性

suppose $\underline{L^* \varphi = e_h}$. and $|\varphi|_{H^2} \leq C \|e_h\|_{L^2}$.

$\|u-u_h\|^2_{L^2(\Omega)} = \|e_h\|^2_{L^2(\Omega)} = (e_h, e_h) = (e_h, L^* \varphi) = (Le_h, \varphi)$

$= a(e_h, \varphi)$

$a(u-u_h, \varphi_h) = 0 \rightarrow$ $= a(e_h, \varphi - \varphi_h)$

$\leq C \|e_h\|_V \|\varphi - \varphi_h\|_V$

$\leq \ldots$ (see P27)

$\leq C h^2 |u|_{H^2} \|e_h\|_{L^2(\Omega)}$.

4. Lemma

$$|\hat{v}|_{m,p,\hat{K}} \leq C \|B\|^m |\det B|^{-\frac{1}{p}} |v|_{m,p,K}, \quad \forall v \in W^{m,p}(K)$$

$$|v|_{m,p,K} \leq C \|B^{-1}\|^m |\det B|^{\frac{1}{p}} |\hat{v}|_{m,p,\hat{K}}, \quad \forall \hat{v} \in W^{m,p}(\hat{K})$$

Idea:

since $\|B\|^m |\det(B)|^{-\frac{1}{p}} \leq h^m \cdot h^{-\frac{n}{p}} = h^{m-\frac{n}{p}}$, then

$$|\hat{v}|_{m,p} \leq C h^{m-\frac{n}{p}} |v|_{m,p}$$

$\|B^{-1}\|^m |\det(B)|^{\frac{1}{p}} \leq h^{-m} \cdot h^{\frac{n}{p}} = h^{-(m-\frac{n}{p})}$, then

$$|v|_{m,p} \leq C h^{-(m-\frac{n}{p})} |\hat{v}|_{m,p}.$$

$$\Rightarrow c_1 |\hat{v}|_{m,p} \leq h^{m-\frac{n}{p}} |v|_{m,p} \leq c_2 |\hat{v}|_{m,p}.$$

Lemma:

$$\|\hat{v}\|_{0,p,\partial\hat{\Omega}} \leq C\|B\|^{\frac{1}{p}}\|B^{-1}\|^{\frac{n}{p}}\|v\|_{0,p,\partial\Omega} \quad \forall v\in L^p(\partial\Omega)$$ "on Boundary".

$$\|v\|_{0,p,\partial\Omega} \leq C\|B^{-1}\|^{\frac{1}{p}}\|B\|^{\frac{n}{p}}\|\hat{v}\|_{0,p,\partial\hat{\Omega}} \quad \forall \hat{v}\in L^p(\partial\hat{\Omega})$$

Idea: $\|B\|^{\frac{1}{p}}\|B^{-1}\|^{\frac{n}{p}} \leq h_K^{\,0-\frac{n-1}{p}}$ ($m=0$) $\to$ n-dimension 在边界上为 $n-1$ 维.

5. Index $m-\frac{n}{p}$:

① For $\|u\|_{L^p} \leq \|u\|_{L^1}^{\theta}\|u\|_{L^\infty}^{1-\theta}$

Let $0-\frac{n}{p} = (0-\frac{n}{1})\theta + (0-\frac{n}{\infty})(1-\theta) \Rightarrow \theta = \frac{1}{p}$.

② For $\|u\|_{W^{k+s,p}} \leq C\|u\|_{W^{k,p}}^{\theta}\|u\|_{W^{k+1,p}}^{1-\theta}$

Let $k+s-\frac{n}{p} = (k-\frac{n}{p})\theta + (k+1-\frac{n}{p})(1-\theta) \Rightarrow \theta = 1-s$.

Class 5

Recall 5. Finite element interpolation theory.

① Global & local FE interpolation operators I_h & $I_{h,K}$

② 1-d polynomial interpolation and Lagrange polys.

③ Higher dimension polynomial interpolation and FE error estimates

(i) Scaling argument
(change of variables by affine mappings)

(ii) Bramble - Hilbert Lemma

(iii)* Inverse inequality for FE functions

$$\left(\text{e.g. } \|\nabla v_h\|_{L^2(K)} \le C h_K^{-1} \|v_h\|_{L^2(K)} \quad \forall\ v_h \in P_r(K) \right)$$

④ Error estimates

(i) Energy norm estimates (By Ceá Lemma)

(ii) L^2-norm estimates (By Duality argument / Aubin-Nitsche)

(iii) L^∞-norm estimates (By weighted Sobolev spaces).

6. Extensions

① To extend Lax-Milgram framework.

② To use nonconforming spaces ($V_h \not\subset V$).

①. A more general weak/variational framework (Extend ①)

H: Hilbert space with $(\cdot,\cdot)_H$

$U, V \subset H$: Banach space with $\|\cdot\|_U$, $\|\cdot\|_V$.

$a(\cdot,\cdot)$: Bilinear form on $U \times V$

$F(\cdot)$: Linear functional on V (i.e., $F \in V^*$).

Consider the following problem:

$$(\text{EGP}) \begin{cases} \text{Find } u \in U \text{ such that} \\ \quad a(u,v) = F(v), \quad \forall\ v \in V \end{cases}$$

Q: Existence & Uniqueness?

Try: (i) Continuity $\quad |a(w,v)| \le C\|w\|_U \|v\|_V \quad \forall w \in U, v \in V.$

(ii) Coercivity (?) $\quad a(v,v)$ does not sense!

(iii) F is continuous/bounded $\quad$ i.e. $|F(v)| \le \|F\|_* \|v\|_V \quad \forall v \in V$

(Weak)
• Generalized coercivity: (a)+(b).

(a) $$\sup_{\substack{v \in V \\ v \ne 0}} \frac{a(w,v)}{\|v\|_V} \ge \alpha \|w\|_U \qquad \forall w \in U. \quad , (\alpha > 0).$$

OR $$\left[\inf_{w \in U, w \ne 0} \sup_{\substack{v \in V \\ v \ne 0}} \frac{a(w,v)}{\|w\|_U \|v\|_V} \ge \alpha \right]$$ Inf-sup Condition

(b) $\sup_{w \in U} a(w, v) > 0 \qquad \forall\, v \in V$ and $v \neq 0$

Theorem (Banach – Nirenberg – Nečas – Babuška)

Assume (i) & (iii), then (EGP) has a unique solution if and only if (a) & (b) hold.

(i.e. generalized coercivity holds).

Moreover, $\|u\|_U \leq \dfrac{\|F\|_*}{\alpha}.$

proof: See Babuška and Aziz's paper (1971)

To prove the stability estimate,

$$a(u, v) = F(v) \qquad (ii)$$

$$\Longrightarrow \alpha \|u\|_U \underset{(a)}{\leq} \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u, v)}{\|v\|_V}$$

$$\underset{(ii)}{=} \sup \frac{F(v)}{\|v\|_V}$$

$$= \|F\|_*$$

Hence $\|u\|_U \leq \dfrac{\|F\|_*}{\alpha}.$

Abstract Petrov – Galerkin method.

Let $\begin{cases} U_m \subset U & (m = \dim U_M) \\ V_n \subset V & (N = \dim V_N) \end{cases}$

Define P-G method as

$$(EGP)_N^M \quad \begin{cases} \text{Find } u_M \in U_M \text{ s.t.} \\ a(u_M, v_N) = F(v_N) \quad \forall\, v_N \in V_N. \end{cases}$$

Q: Existence of Uniqueness & Convergence ?

Theorem: (Generalized Ceá Lemma)

Suppose $a(\cdot,\cdot)$ and $F(\cdot)$ satisfy (i), (ii) & ⓐ+ⓑ (Hence, (EGP) has a unique solution).

Assume $\exists\, \alpha_M > 0$ such that

(even if $\begin{cases} V_N \subset V \\ U_M \subset U \end{cases}$, 连续的 ⓐ+ⓑ do not imple 离散的 ⓐ'+ⓑ'.)

Discrete Weak Coercivity

ⓐ' $\displaystyle \sup_{v_N \in V_N} \frac{a(w_M, v_N)}{\|v\|_V} \geq \alpha_M \|w_M\|_{U_M}, \quad \forall\, w_M \in U_M \subset U.$

ⓑ' $\displaystyle \sup_{w_M \in U_M} a(w_M, v_N) > 0 \quad \forall\, v_N \in V_N,\ v_N \neq 0.$

Then $(EGP)_N^M$ has a unique solution $w_M \in U_M$.

Moreover, $\|u - u_M\|_U \leq \left(1 + \dfrac{C}{\alpha_M}\right) \|u - w_M\|_U \qquad \forall\, w_M \in U_M.$

proof : <u>part 1</u> $\{\varphi_j\}_{j=1}^{N}$: basis for V_N

$\{\psi_i\}_{i=1}^{M}$: basis for U_M

write $u_M = \sum_{j=1}^{M} \xi_j \psi_j$, Then $(EGP)_N^M$ can be equivalently written as

$$\underline{A\vec{\xi} = \vec{b}} \quad N\times M \text{ rectangular linear system.}$$

where $A = [a_{ij}]_{N\times M}$, $a_{ij} = a(\psi_j, \varphi_i)$

$\vec{b} = [b_i]_{1\times N}$, $b_i = F(\varphi_i)$.

$\vec{\xi} = [\xi_j]_{1\times M} = [\xi_1, \dots \xi_M]^T$,

<u>Claim</u> (Exercise)

ⓐ$'$ + ⓑ$'$ imply that $\underset{\substack{\uparrow \\ \text{minimum singular value of } A}}{\sigma_{min}}(A) \geq \alpha_h \ (h>0)$

Hence, A is of full rank and $A\vec{\xi} = \vec{b}$ has a unique solution.

<u>part 2</u> Set $e_M := u - u_M = \underbrace{(u - w_M)}_{=:\eta_M} + \underbrace{(w_M - u_M)}_{+\ \xi_M \in U_M} \quad \forall w_M \in U_M.$

we have $a(e_M, v_N) = 0$. (Galerkin orthogonality).

since

$$\alpha_h \|\xi\|_U \overset{ⓐ'}{\leq} \sup_{v_N\in V_N} \frac{a(\xi_M, v_N)}{\|v_N\|_V} = \sup_{v_N\in V_N} \frac{\overset{0}{\cancel{a(e_M, v_N)}} - a(\eta_M, v_N)}{\|v_N\|_V}$$

$$= \sup_{v_N\in V_N} \frac{-a(\eta_M, v_N)}{\|v_N\|_V} \underset{(i)}{\leq} C\|\eta_M\|_U.$$

then $\|e_M\|_U \leq \|\eta_M\|_U + \|\xi_M\|_U$

$$= \left(1 + \frac{C}{\alpha_h}\right)\|\eta_M\|_U$$

<u>Remark</u> : Galerkin Orthogonality plays a key role in the proof

$$a(u_M, v_N) = a(u, v_N) \quad \forall v_N \in V_N.$$

<u>Definition</u> : u_M is called an " elliptic " projection of u.

Petrov - Galerkin FE Method.

Set $V_N = V_h$, $U_M = U_h$, V_h & U_h are two FE spaces associated with mesh $\mathcal{T}_h$.

Then P-G method gives PG- FE method.

$$(EPG)_h. \begin{cases} \text{Find } u_h \in U_h \text{ s.t.} \\ a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \end{cases}$$

Hence, the energy norm error estimate follows from combining Generalized Céa lemma & FE interpolation theory.

② Nonconforming Galerkin/FE methods

Only consider Galerkin method.

since Petrov-Galerkin method can be done similarly $\left(\begin{matrix} V_N = U_M \\ V = U \end{matrix}\right)$.

$V_N \not\subset V$ (Nonconforming).

Main issues / difficulties

(i) $a(\cdot,\cdot)$ and $F(\cdot)$ may not be defined on $V_N \times V_N$ and V_N.

(ii) Galerkin orthogonality may not hold.

(iii) Must deal with mesh-dependent norms.

$\|\cdot\|_{V_h}$

Setting (i) $a_N(\cdot,\cdot)$: bilinear form $(V_N \cup V) \times (V_N \cup V)$

$F_N(\cdot)$: linear functional on $V_N \cup V$.

(ii) Galerkin method now become

$$(NC\text{-}GP)_N \begin{cases} \text{Find } u_N \in V_N \text{ s.t.} \\ a_N(u_N, v_N) = F_N(v_N) \quad \forall v_N \in V_N \end{cases}$$

Definition $(NC\text{-}GP)_N$ is said to be consistent if

$$a_N(\underset{\uparrow}{u}, v_N) = F_N(v_N) \qquad \forall v_N \in V_N$$

exact solution of EGP

Otherwise, it is said to be inconsistent. with respect to (EGP).

Remark: All conforming Galerkin methods are consistent methods.

Fact All consistent methods satisfy the Galerkin orthogonality

$$a_N(u-u_N, v_N) = 0 \qquad \forall v_N \in V_N$$

which implies Ceá Lemma.

Remark: For any inconsistent method, the Galerkin orthogonality does not hold.

$$\underbrace{a(u-u_N, v_N)}_{\text{inconsistent error.}} \neq 0$$

Theorem: (Strong 2nd Lemma) $\longrightarrow$ (in place of Ceá Lemma).

Suppose that (i) $|a_N(w_N, v_N)| \le C \|w_N\|_{V_N} \|v_N\|_{V_N}$, $\forall w_N, v_N \in V_N \cup V$

(ii) $a_N(v_N, v_N) \ge \alpha \|v_N\|^2_{V_N}$, $\forall v_N \in V_N \cup V$

(iii) $|F(v_N)| \le M \|v_N\|_{V_N}$, $\forall v_N \in V_N \cup V$.

Then $\exists!$ $u_N \in V_N$ s.t.

(a) $\|u_N\|_V \le \frac{M}{\alpha}$

(b) $\|u-u_N\|_{V_N} \le \underbrace{\left(1+\frac{C}{\alpha}\right) \inf_{v_N \in V_N} \|u - v_N\|_{V_N}}_{\text{Approximation error}} + \underbrace{\frac{1}{\alpha} \sup_{v_N \in V_N} \frac{a(u-u_N, v_N)}{\|v_N\|_{V_N}}}_{\text{inconsistency error.}}$

proof of ⓑ: Set $e_N := u - u_N = (u - v_N) + (v_N - u_N) =: \eta_N + \xi_N$ $(\xi_N \in V_N)$. $\forall v_N \in V_N$

Since $\alpha \|\xi_N\|_{V_N}^2 \underset{(ii)}{\leq} a(\xi_N, \xi_N) = a(e_N - \eta_N, \xi_N)$

$$= a(e_N, \xi_N) - a(\eta_N, \xi_N)$$

$$\leq a(e_N, \xi_N) + c \|\eta_N\|_{V_N} \cdot \|\xi_N\|_{V_N}.$$

then

$$\|\xi_N\|_{V_N} \leq \frac{c}{\alpha} \|\eta_N\|_{V_N} + \frac{1}{\alpha} \frac{a(e_N, \xi_N)}{\|\xi_N\|_{V_N}}$$

By the triangle inequality

$$\|e_N\|_{V_N} \leq \|\eta_N\|_{V_N} + \|\xi_N\|_{V_N} \leq \left(1 + \frac{c}{\alpha}\right) \|\xi_N\|_{V_N} + \frac{1}{\alpha} \frac{a(e_N, \xi_N)}{\|\xi_N\|_{V_N}}$$

$$\leq \left(1 + \frac{c}{\alpha}\right) \|\xi_N\|_{V_N} + \frac{1}{\alpha} \sup_{v_N \in V_N} \frac{a(e_N, \;)}{\|v_N\|_{V_N}}$$

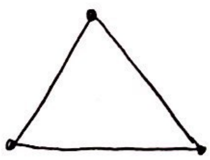
Take $\inf_{v_N \in V_N}$ on both sides yields the desired inequality.

Example 1. $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$V = H_0^1(\Omega)$ $\quad a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$

Given $\mathcal{T}_h$ (triangular mesh)

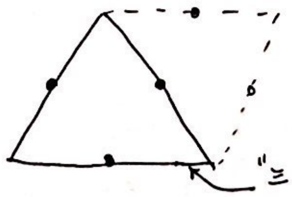
$$V_1(K) = P_1(K), \text{ where } \dim(P_1(K)) = \frac{(d+r)!}{d! \, r!} = \frac{(1+2)!}{1! \, 2!} = 3.$$



P_1 - conforming

$\Downarrow$

V_h



"三中点元"

P_1 - nonconforming

$\Downarrow$

$$V_h^{NC} := \left\{ \begin{array}{l} v_h : \Omega \to \mathbb{R};\ v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h, \\ v_h \text{ is continuous at all interior midpoints of } \mathcal{T}_h \end{array} \right\}$$

Fact: $V_h^{NC} \not\subset C^0(\bar{\Omega}) \Rightarrow V_h^{NC} \not\subset H^1(\Omega)$.

Exercise: prove the above conclusion

$$a_h(w_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K \nabla w_h \cdot \nabla v_h \, dx$$

$$F_h(v_h) := \sum_{K \in \mathcal{T}_h} \int_K f \cdot v_h \, dx = \int_\Omega f v \, dx = F(v).$$

$$\|v_h\|_{V_h^{NC}} := \left[a_h(v_h, v_h) \right]^{\frac{1}{2}}.$$

P_1 - nonconforming FE Method.

$$\begin{cases} \text{Find } u_h \in V_h^{NC} \text{ s.t.} \\ a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h^{NC}. \end{cases}$$

Theorem : $\| u - u_h \|_{V_h^{NC}} \le C h |u|_{H^2(\Omega)}$ (By strong 2^{nd} lemma)

$V = "H_0^1" \to V_h^{NC}$

$\| u - u_h \|_{L^2(\Omega)} \le C h^2 |u|_{H^2(\Omega)}$ (By Duality argument).

Remark : $V_h^{NC} \supset V_h := V_1^h$

Example 2. IP-DG (interior-penalty DG) Method

Given $\mathcal{T}_h$ over Ω, $V_h^{DG} = \{ v_h : \Omega \to R \,;\, v_h|_K \in P_r(K),\ \forall K \in \mathcal{T}_h \}$

$$= \prod_{K \in \mathcal{T}_h} P_r(K)$$

The central issue of IP-DG methods is how to construct discrete bilinear form $a_h(\cdot,\cdot)$.

IV. Application

Comment : FEM is a generic method which can be used to solve any application problem which is described by PDE(s).

1. Linear elasticity (stationary case)

Lamé system $\begin{cases} -\operatorname{div} \sigma(\vec{u}) = \vec{f} & \text{in } \Omega \subset R^d \\ \vec{u} = 0 & \text{on } \partial\Omega \end{cases}$ $\quad \vec{u}$: displacement vector.

where $\begin{cases} \sigma(\vec{u}) = \lambda \operatorname{div} \vec{u}\, I_{d \times d} + \mu\, \varepsilon(\vec{u}). & \text{(stress tensor)}. \\ \varepsilon(\vec{u}) = \frac{1}{2} \left(\nabla \vec{u} + (\nabla \vec{u})^T \right) & \text{(strain tensor)}. \end{cases}$

Weak formulation

$$\int_\Omega \underbrace{\sigma(\vec{u})\, \varepsilon(\vec{v})}_{\lambda \operatorname{div} \vec{u} \cdot \operatorname{div} \vec{v} + \mu\, \varepsilon(\vec{u}) : \varepsilon(\vec{v}).} dx = \int_\Omega f \vec{v}\, dx \quad \forall \vec{v} \in \underbrace{[H_0^1(\Omega)]^d}_{V}$$

Remark : In particular, since $\sigma(\vec{u})$ is more important than $\vec{u}$, which should be approximated seperately. $\Rightarrow$ Mixed FE method.
$(\vec{u}, \sigma)$.

2. Incompressible fluids (stationary case).

Stokes equations
$$\begin{cases} -\nu \Delta \vec{u} + \nabla p = \vec{f} & \text{in } \Omega \subset \mathbb{R}^d \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega \\ \vec{u} = 0 & \text{on } \partial\Omega \end{cases}$$

$\vec{u} := \vec{u}(x,t)$: velocity of fluid at (x,t)

$p := p(x,t)$ pressure of fluid at (x,t).

Navier-Stokes equations
$$\begin{cases} -\nu \Delta \vec{u} + \underbrace{\boxed{(\vec{u}\cdot\nabla)\vec{u}}}_{} + \nabla p = \vec{f} & \text{in } \Omega \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega \\ \vec{u} = 0 & \text{on } \partial\Omega \end{cases}$$

Nonlinear

① Choose $\tilde{V} = \{ \vec{v} \in [H_0^1(\Omega)]^d ; \operatorname{div} \vec{v} = 0 \}$

then weak form is

$$\nu \int_\Omega \nabla\vec{u} \cdot \nabla\vec{v}\, dx + \int_{\partial\Omega} p \cdot \underbrace{\cancel{\vec{v}\cdot\vec{n}}}_{=0}\, ds - \int_\Omega p\, \underbrace{\cancel{\operatorname{div}\vec{v}}}_{=0}\, dx = \int_\Omega \vec{f}\cdot\vec{v}\, dx, \ \forall v \in \tilde{V}.$$

$$\Rightarrow \quad \nu(\nabla\vec{u}, \nabla\vec{v}) = (\vec{f}, \vec{v}) \quad \forall \vec{v} \in \tilde{V}.$$

Remark: Only $\vec{u}$ is approximated.

② Mixed Weak formulation.

Let $V = [H_0^1(\Omega)]^d$

$W = L_0^2(\Omega) = \{ q \in L^2(\Omega) ; \int_\Omega q\, dx = 0 \}$.

then weak formulation is

$$\begin{cases} \overbrace{\nu(\nabla\vec{u}, \nabla\vec{v})}^{a(\vec{u},\vec{v})} - \overbrace{(p, \operatorname{div}\vec{v})}^{b(p,\vec{v})} = (\vec{f}, \vec{v}) & \forall \vec{v} \in V \\ \underbrace{(\operatorname{div}\vec{u}, q)}_{b(q,\vec{u})} = 0 \ \ (\text{or } (g,q)) & \forall q \in W. \end{cases}$$

$$\Rightarrow \circledast \begin{cases} a(\vec{u},\vec{v}) + b(\vec{v},p) = (\vec{f},\vec{v}) & \forall \vec{v} \in V \\ b(q,\vec{u}) = 0 & \forall \end{cases}$$

Define $\mathcal{L}((\vec{u},p),(\vec{v},q)) := a(\vec{u},\vec{v}) - b(\vec{v},p) + b(\vec{u},q)$, on $(V\times W)\times(V\times W)$

where $\forall (\vec{u},p) \in/ (\vec{v},q) \in (V\times W)\times(V\times W)$.

$\mathcal{F}((\vec{v},q)) := (\vec{f},\vec{v}) + (g,q)$. on $V\times W$

Fact: $\circledast$ is equivalent to

$$(\ast\ast) \quad \mathcal{L}((\vec{u},p),(\vec{v},q)) = \mathcal{F}((\vec{v},q)), \ \forall (\vec{v},q) \in V\times W.$$

Fact: $\mathcal{L}$ is not coercive on $V\times W$. Hence, Lax-Milgram does not apply to $(\ast\ast)$. But it can be shown that $\mathcal{L}(\cdot,\cdot)$ is weakly coercive (i.e. $\mathcal{L}(\cdot,\cdot)$ satisfies ⓐ&ⓑ). Hence, Banach-Nirenberg-Necas-Babuska Thm Applies to $(\ast\ast)$! — Under some conditions. [inf-sup condition on $b(\cdot,\cdot)$].