Examination – Numerical PDEs, Spring 2018

Due on June 8th, 2018

1. Solve the following initial value problems analytically:

   (a) $y' = t^3$; $y(0) = 0$.

   (b) $y' = ay + b$; $y(0) = y_0$, where a and b are scalars.

   (c) $y' = a(1 - y^2)$; $y(0) = 0$, where a is a scalar.

2. Find the range of α for which the linear multistep method

$$y_{n+3} + \alpha(y_{n+2} - y_{n+1}) - y_n = \frac{1}{2}(3+\alpha)h\left(f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1})\right)$$

   is zero-stable. Show that there is a value of α for which the method has order 4 but that if the method is to be zero-stable, its order cannot exceed 2.

3. Show that the following scheme is consistent with the equation $u_t + cu_{tx} + au_x = f$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1} - u_{j+1}^n + u_{j-1}^n}{2\Delta t\Delta x} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = f_j^n.$$

4. Show that the following scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+2}^n - 3u_{j+1}^n + 3u_j^n - u_{j-1}^n}{(\Delta x)^3} = f_j^n$$

   is consistent with $u_t + au_{xxx} = f$. Moreover, if $s = a\Delta t/(\Delta x)^3$ is constant, then it is stable when $0 \le s \le 1/4$.

5. Show that the solution of the initial value problem for

$$u_t + u_x = \cos^2 u$$

   is given by

$$u(x,t) = \tan^{-1}\{\tan[u_0(x - t)] + t\},$$

   where $u_0(x)$ is the initial condition.

6. Show that the following scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2}\{\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} + \frac{u_j^n - u_{j-1}^n}{\Delta x}\} = f_j^n$$

is consistent with

$$u_t + au_x = f$$

7. Find an differential equation where the following different scheme is consistent with

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{1}{2}\frac{\Delta t}{(\Delta x)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n).$$

When will the scheme be stable?

8. Consider a scheme for $u_t = u_{xx}$ of the form

$$u_j^{n+1} = (1 - 2\alpha - 2\beta)u_j^n + \alpha(u_{j+1}^n + u_{j-1}^n) + \beta(u_{j+2}^n + u_{j-2}^n).$$

Denote $\mu = \Delta t/(\Delta x)^2$. Show that when μ is a constant, as Δt and Δx tend to zero, that the scheme is inconsistent unless

$$\alpha + 4\beta = \mu.$$

Show that the scheme is four-order accurate in x if $\beta = -\alpha/16$.