Hw♯6–The multi-dimensional case for periodic composite materials

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October 23, 2017

I will present a multi-dimensional problem in periodic composite materials. Let Ω denotes a bounded open set in R^N and $\epsilon > 0$ is a parameter taking its values in a sequence which tends to zero. Let

$$A^\epsilon(x) = (a^\epsilon_{ij}(x))_{1\leq i,j\leq N}, \quad \text{a.e. on } \Omega, \tag{1}$$

be a sequence suffices to

$$A^\epsilon \in M(\alpha, \beta, \Omega), \tag{2}$$

i.e.,

$$\begin{aligned}(A^\epsilon\lambda, \lambda) \geq \alpha|\lambda|^2 \\ |A^\epsilon\lambda| \leq \beta|\lambda|,\end{aligned}$$

for any $\lambda \in R^N$ and a.r. on Ω, and $A^\epsilon \in L^\infty(\Omega)$.

Consider the Dirichlet problem

$$\begin{cases} -div(A^\epsilon\nabla u^\epsilon) = f \text{ in } \Omega \\ u^\epsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where f is given in $H^{-1}(\Omega)$.

Introduce the operator

$$\mathcal{A}_\epsilon = -div(A^\epsilon\nabla) = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a^\epsilon_{ij}\frac{\partial}{\partial x_j}\right). \tag{4}$$

Then, we need to solve this system

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a^\epsilon_{ij}\frac{\partial u^\epsilon}{\partial x_j}\right) = f \text{ in } \Omega \\ u^\epsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{5}$$

As we all know, set

$$Y = [0, \ell_1] \times [0, \ell_2] \times \cdots \times [0, \ell_N],$$

where $\ell_1, \ell_2, \cdots, \ell_N$ are given positive numbers. It is called the reference period or cell.

We assume here that a_{ij} is positve function in $L^\infty(0, \ell_1)$ such that

$$\begin{cases} a_{ij} \text{ is } Y-\text{peridic}, \forall i,j = 1, \cdots, N \\ 0 < \alpha \leq a_{ij}(x) \leq \beta < +\infty, \end{cases} \tag{6}$$

where $\alpha, \beta \in R$, and both are positive. Note that

$$a_{ij}^\epsilon = a_{ij}\left(\frac{x}{\epsilon}\right) \text{ a.e. on } R^N, \forall i,j = 1, \cdots, N, \tag{7}$$

and

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right) = (a_{ij}^\epsilon(x))_{1\leq i,j\leq N} \text{ a.e. on } R^N. \tag{8}$$

Theorem 1 *(Homogenization Direchlet problem) Suppose taht the matrix A belongs to $M(\alpha, \beta, \Omega)$. Then, for any $f \in H^{-1}(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ of the variational problem*

$$\begin{cases} \textit{Find } u \in H_0^1(\Omega) \textit{ such that} \\ a(u,v) =< f, v >_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{9}$$

where

$$a(u,v) = \sum_{i,j=1}^N \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega A \nabla u \nabla v \, dx. \tag{10}$$

Moreover,

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}, \tag{11}$$

where $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2\Omega}$.

If $f \in L^2(\Omega)$, the solution satisfies the estimate

$$\|u\|_{H_0^1(\Omega)} \leq \frac{C_\Omega}{\alpha}\|f\|_{L^2(\Omega)}, \tag{12}$$

where C_Ω is the Poincaré constant.

From Theorem 1, it follows that for any fixed ϵ, there exists a unique solution $u^\epsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega A^\epsilon \nabla u^\epsilon \nabla v \, dx =< f, v >_{H^{-1}(\Omega), H_0^1(\Omega)}, \forall v \in H_0^1(\Omega). \tag{13}$$

Moreover, one has

$$\|u^\epsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{14}$$

Theorem 2 *(Eberlein-Šmuljan) Assume that E is reflective and let x_n be a bounded sequence in E. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that, as $k \to \infty$,*

$$x_{n_k} \rightharpoonup x \text{ weakly in } E.$$

Theorem 3

- *The space $W^{1,p}(O)$ is a Banach space for the norm*

$$\|u\|_{W^{1,p}(O)} = \|u\|_{L^p(O)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^p(O)}.$$

For $1 \leq p < \infty$, this norm is equivalent to the following one,

$$\|u\|_{W^{1,p}(O)} = \left(\|u\|^p_{L^p(O)} + \|\nabla u\|^p_{L^p(O)}\right)^{\frac{1}{p}}, \tag{15}$$

where we have useed the notations

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right).$$

and

$$\|\nabla u\|_{L^p(O)} = \left(\sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|^p_{L^p(O)}\right)^{\frac{1}{p}}.$$

- *The space $W^{1,p}(O)$ is separable for $1 \leq p < +\infty$ and reflective for $1 < p < +\infty$.*
- *The space $H^1(O)$ is a Hilbert space for the scalar product*

$$(v, w)_{H^1(O)} = (v, w)_{L^2(O)} + \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i}\right)_{L^2(O)}, \quad \forall v, w \in H^1(O). \tag{16}$$

From Thm. 2 and Thm. 3, it follows that there exists a subsequence $\{u^{\epsilon'}\}$ and an element $u^0 \in H^1_0(\Omega)$ such that

$$u^{\epsilon'} \rightharpoonup u^0 \text{ weakly in } H^1_0(\Omega).$$

Let me introduce the vector

$$\xi^\epsilon = (\xi^\epsilon_1, \cdots, \xi^\epsilon_N) = \left(\sum_{j=1}^{N} a^\epsilon_{1j}\frac{\partial u^\epsilon}{\partial x_j}, \cdots, \sum_{j=1}^{N} a^\epsilon_{Nj}\frac{\partial u^\epsilon}{\partial x_j}\right) = A^\epsilon \nabla u^\epsilon, \tag{17}$$

which satisfies

$$\int_\Omega \xi^\epsilon \nabla v \, dx = < f, v >_{H^{-1}(\Omega), H^1_0(\Omega)}, \forall v \in H^1_0(\Omega). \tag{18}$$

From $A^\epsilon \in M(\alpha, \beta, \Omega)$ and (14), it follows that

$$\|\xi^\epsilon\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha} \|f\|_{H^{-1}(\Omega)}. \tag{19}$$

Again from Thm.2, there exists a subsequence, still denotedd by $\{\xi^{\epsilon'}\}$, and an element $\xi^0 \in L^2(\Omega)$, such that

$$\xi^{\epsilon'} \rightharpoonup \xi^0 \text{ weakly in } (L^2(\Omega))^N. \tag{20}$$

Hence, we can pass to the limit in 18 writen for the subsequence ϵ', to get

$$\int_\Omega \xi^0 \nabla v \, dx =< f, v >_{H^{-1}(\Omega), H_0^1(\Omega)}, \, \forall v \in H_0^1(\Omega), \tag{21}$$

i.e.,

$$-div(\xi^0) = f \text{ in } \Omega.$$

Theorem 4 *(Weak limits of rapidly oscillating periodic functions) Let $a \leq p \leq +\infty$ and f be a $Y-$periodic function in $L^p(Y)$. Set*

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \; a.e. \; on \; R^N. \tag{22}$$

Then, if $p < +\infty$, as $\epsilon \to 0$,

$$f_\epsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|} \int_Y f(y)\, dy \text{ weakly in } L^p(\omega),$$

for any bounded open subset ω of R^N.

If $p = +\infty$, one has

$$f_\epsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|} \int_Y f(y)\, dy \text{ weakly* in } L^\infty(R^N).$$

Obeserve that from Thm.4, it follows that if $\epsilon \to 0$,

$$A^\epsilon \rightharpoonup \mathcal{M}_Y(A) \text{ weakly* in } L^\infty(\Omega), \tag{23}$$

where the matrix $(\mathcal{M}_Y(A))_{ij}$ is defined by

$$(\mathcal{M}_Y(A))_{ij} = \frac{1}{|Y|} \int_Y a_{ij}(y)\, dy. \tag{24}$$

As we all know, $A^\epsilon \nabla u^\epsilon$ is the product of two weakly convergent sequences. But in general,

$$\xi^0 \neq \mathcal{M}_Y(A) \nabla u^0. \tag{25}$$

Since the coeffiicients of A^0 are no longer obtained as algebra formulas from A, for the general $N-$dimensional case, the situation is different from the $1-$dimentional case.

In order to study the general $N-$dimensional case, we need to introduce some auxiliary functions which are solutions of periodic boundary value problem in the reference cell Y. In the sequel, we will state the asymptotic behaviour as $\epsilon \to 0$.

We will take advantage of the two kind of operators, ones is $\mathcal{A} = -div(A\nabla)$, the functions introduced are $\hat{\chi}_\lambda$ and $\hat{\omega}_\lambda$, the other is $\mathcal{A}^* = -div(A^T\nabla)$, the functions introduced are χ_λ and ω_λ.

Consider the solutions of system

$$\begin{cases} -div(A(y)\nabla\hat{\chi}_\lambda) = -div(A(y)\lambda) \text{ in } Y \\ \hat{\chi}_\lambda \text{ Y-periodic} \\ \mathcal{M}_Y(\hat{\chi}_\lambda) = 0, \end{cases} \tag{26}$$

and system

$$\begin{cases} -div(A^T(y)\nabla\chi_\lambda) = -div(A^T(y)\lambda) \text{ in } Y \\ \chi_\lambda \text{ Y-periodic} \\ \mathcal{M}_Y(\chi_\lambda) = 0, \end{cases} \tag{27}$$

we can write the variational formulation of the two system and do its extension by periodicity to the whole R^N, then, we take ω_λ to the new problem which solved as previously.

Theorem 5 *(convergence) Let $f \in H^{-1}(\Omega)$ and u^ϵ be the solution of (3), then, one has*

$$\begin{cases} i)\ u^\epsilon \rightharpoonup u^0 \ \textit{weakly in } H_0^1(\Omega), \\ ii)\ A^\epsilon\nabla u^\epsilon \rightharpoonup A^0\nabla u^0 \ \textit{weakly in } (L^2(\Omega))^N, \end{cases} \tag{28}$$

where u^0 is the unique solution in $H_0^1(\Omega)$ of the homogenized system

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{i,j}^0 \frac{\partial u^0}{\partial x_j}\right) = f \ \textit{in } \Omega, \\ u^0 = 0 \ \textit{on } \partial\Omega. \end{cases} \tag{29}$$

The matrix $A^0 = (a_{ij}^0)_{1\le i,j\le N}$ is constant, elliptic and given by

$$A^0\lambda = \mathcal{M}_Y(A\nabla\hat{\omega}_\lambda) \quad \forall\lambda \in R^N, \tag{30}$$

i.e.,

$${}^tA^0\lambda = \mathcal{M}_Y({}^tA\nabla\omega_\lambda) \quad \forall\lambda \in R^N, \tag{31}$$

Theorem 6 *Let $f \in H^{-1}$ and u^ϵ be the solution of (3). Then, u^ϵ admits the following asymptotic expansion*

$$u^\epsilon = u_0 - \epsilon\sum_{k=1}^N \hat{\chi}_k\left(\frac{x}{\epsilon}\right) + \epsilon^2 \sum_{k,\ell=1}^N \hat{\theta}^{kl}\left(\frac{x}{\epsilon}\right)\frac{\partial^2 u_0}{\partial x_k \partial x_l} + \cdots$$

where u_0 is solution of (29), $\hat{\chi}_k \in W_{per}(Y)$ and $\hat{\theta}^{k\ell}$ by

$$\begin{cases} -div(A(y)\nabla\hat{\theta}^{k\ell}) = -a^0_{k\ell} - \sum_{i,j=1}^N \frac{\partial(a_{ij}\delta_{ki}\hat{\chi}_\ell)}{\partial y_i} - \sum_{j=1}^N a_{kj}\frac{\partial(\hat{\chi}_\ell - y_\ell)}{\partial y_j} \text{ in } Y, \\ \hat{\theta}^{k\ell} \text{ } Y\text{-periodic}, \\ \mathcal{M}_Y(\hat{\theta}^{k\ell}) = 0. \end{cases}$$

Moreover, if $f \in C^\infty(\Omega)$, $\partial\Omega$ is of class C^∞ and

$$\hat{\chi}_k, \hat{\theta}^{k\ell} \in W^{1,\infty}(Y), \quad \forall k, \ell = 1, \cdots, N$$

then, there exists a constant C independent of ϵ, such that

$$\|u^\epsilon - \left(u_0 - \epsilon \sum_{k=1}^N \hat{\chi}_k\left(\frac{x}{\epsilon}\right) + \epsilon^2 \sum_{k,\ell=1}^N \hat{\theta}^{kl}\left(\frac{x}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \|_{H^1(\Omega)} \leq C\epsilon^{1/2}.$$