Mathematics for Material Sciences **2017/05/01**

Note 3. The Energy Minimization to Deduce The Landau-Lifhiz Equation

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Consider a non-dimensionalization of the Landau-Lifshitz equation, which will be proved in what follows using energy minimization. The Landau-Lifshitz energy can be written in dimensionless variables by rescaling $\mathbf{M} = M_s\mathbf{m}$, $\mathbf{H}_s = M_s\mathbf{h}_s$, $U = M_s u$, $\mathbf{H}_e = M_s\mathbf{h}_e$, $\mathbf{x} = L\mathbf{x}'$, $F[\mathbf{M}] = (\mu_0 M_s^2 L^3)F[\mathbf{m}]$, where $\mathbf{M}$ is magnetization which has units of A/m and and dimensions $[\mathbf{M}] = [M_s] = AL^{-1}$, mathematically, it is a vector field of constant length M_s (in units of A/m), where $\mathbf{H_s}$ is stray field, U is scalar function.

The Laudau-Lifshitz free energy considered is

$$F[\mathbf{m}] = q\int_{\Omega'}\left(m_2^2+m_3^2\right)\,d\mathbf{x}' + \epsilon\int_{\Omega'}|\nabla\mathbf{m}|^2 d\mathbf{x}' + \frac{1}{2}\int_{\mathbb{R}}|\nabla u|^2\,d\mathbf{x}' - \int_{\Omega'}\mathbf{h}_e\cdot\mathbf{m}d\mathbf{x}', \tag{1}$$

where $q = 2K_u/(\mu_0 M_s^2)$ and $\epsilon = 2C_{ex}/(\mu_0 M_s L^2)$ are dimensionless. Note that $\mathbf{m} = \frac{\mathbf{M}}{M_s}$ and $|\mathbf{M}| = M_s$, $\mathbf{m} = (m_1, m_2, m_3)^T$ then one has $|\mathbf{m}| = 1$, we are interested in the problem of minimizing the energy $F[\mathbf{m}]$ over all functions belonging to the admissible calss

$$\mathcal{A} = \{\mathbf{m}\in H^1(\Omega;\mathbb{R}^3)\,|\,\frac{\partial\mathbf{m}}{\partial\nu} = \mathbf{0}\quad\text{on }\ \partial\Omega,\ |\mathbf{m}| = 1\text{ a.e.}\}.$$

In the sequel, the variational calculus is adopted to obtain Landau-Lifschitz equation, that is the necessity of energy minimization, i.e., $\frac{\delta F}{\delta\mathbf{m}} = 0$. Assume $\mathbf{v}\in C_c^\infty(\Omega')$, then since $|\mathbf{m}| = 1$, we give a turbulent and note that $|\mathbf{m}+t\mathbf{v}|\neq 1$, for each sufficiently small t, we consider first the minimization without constraints using energy methods. At beginning of this process, for all $t\in\mathbb{R}$ and $\mathbf{v}\in C_c^\infty(\Omega)$, $I[t] = F[\mathbf{m}+t\mathbf{v}]\geq F[\mathbf{m}]$, it follows $I'[t]|_{t=0} = 0$. We then have

$$\begin{aligned}
I[t] &= F\left[\mathbf{m}+t\mathbf{v}\right]\\
&= q\int_{\Omega'}\left[(m_2+tv_2)^2+(m_3+tv_3)^2\right]\,d\mathbf{x}'\\
&+\epsilon\int_{\Omega'}|\nabla(\mathbf{m}+t\mathbf{v})|^2\,d\mathbf{x}'\\
&+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,d\mathbf{x}'\\
&-\int_{\Omega'}\mathbf{h}_e\cdot(\mathbf{m}+t\mathbf{v})\,d\mathbf{x}'.
\end{aligned}$$

We note that $\mathbf{div}(-\nabla u+\mathbf{m}+t\mathbf{v}) = 0$, for $x\in\mathbb{R}^3$, we take derivative with regard to t, that is, $\mathbf{div}(-\nabla\frac{\partial u}{\partial t}+\mathbf{v}) = 0$, then

$$\int_{\mathbb{R}^3}\nabla(\frac{\partial u}{\partial t})\cdot\nabla u\,dx = \int_\Omega\mathbf{v}\cdot\nabla u\,dx.$$

Taking derivative with respect to variable t on the both side, we obtain

$$\begin{aligned} I'[t] = q &\int_{\Omega'} 2\left[(m_2+tv_2)v_2+(m_3+tv_3)v_3\right]\, d\mathbf{x}' \\ &+ \epsilon \int_{\Omega'} 2\nabla(\mathbf{m}+t\mathbf{v})\cdot\nabla\mathbf{v}\, d\mathbf{x}' \\ &+ \int_{\mathbb{R}^3} \nabla u\cdot\mathbf{v}\, d\mathbf{x}' \\ &- \int_{\Omega'} \mathbf{h}_e\cdot\mathbf{v}\, d\mathbf{x}'. \end{aligned}$$

We observe that for all $x\in\partial\Omega$, it follows $\frac{\partial\mathbf{m}}{\partial\nu}=\mathbf{0}$, and also note that $\nabla\mathbf{m}\cdot\nabla\mathbf{v}=-\Delta\mathbf{m}\cdot\mathbf{v}$, where $\int_{\Omega'}\Delta\mathbf{m}\cdot\mathbf{v}\,dx$ is easy to calculate due to $\mathbf{v}$ decay into 0 on boundary using integrate by parts. Then,

$$\begin{aligned} I'[0] = q &\int_{\Omega'} 2\left[m_2v_2+m_3v_3\right]\, d\mathbf{x}' \\ &- \epsilon \int_{\Omega'} 2\Delta\mathbf{m}\cdot\mathbf{v}\, d\mathbf{x}' \\ &+ \int_{\mathbb{R}^3} \nabla u\cdot\mathbf{v}\, d\mathbf{x}' \\ &- \int_{\Omega'} \mathbf{h}_e\cdot\mathbf{v}\, d\mathbf{x}'. \end{aligned}$$

Thus,

$$\mathbf{h}_{\text{eff}} = 2q(m_2\mathbf{e}_2+m_3\mathbf{e}_3) - 2\epsilon\Delta\mathbf{m} + \nabla u - \mathbf{h}_e = \mathbf{0}.$$

In the following, we consider the minimization with constraints using energy methods. For simplicity, we introduce a notation as follows

$$\gamma(t) = \frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|} \in \mathcal{A}.$$

Thus,

$$I[t] = F\left[\gamma(t)\right] = F\left[\frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}\right].$$

Note, however, that

$$\gamma'(t) = \frac{\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|} - \frac{[(\mathbf{m}+t\mathbf{v})\cdot\mathbf{v}](\mathbf{m}+t\mathbf{v})}{|\mathbf{m}+t\mathbf{v}|^3},$$

At beginning of this process, for all $t\in\mathbb{R}$ and $\mathbf{v}\in C_c^\infty(\Omega)$,$I[t]=F\left[\gamma(t)\right]=F\left[\frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}\right]\geq F[\mathbf{m}]$, it follows $I'[t]|_{t=0}=0$. In the following, we note that

$$\nabla\left(\frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}\right) = \frac{\nabla\mathbf{m}+t\nabla\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|} - \frac{(\mathbf{m}+t\mathbf{v})[(\mathbf{m}+t\mathbf{v})\cdot\nabla(\mathbf{m}+t\mathbf{v})]}{|\mathbf{m}+t\mathbf{v}|^3}.$$

We will give a computation in detail, i.e.,

$$\begin{aligned}
I[t] &= F\left[\frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}\right] \\
&= q\int_{\Omega'}\left[\left(\frac{m_2+tv_2}{|\mathbf{m}+t\mathbf{v}|}\right)^2+\left(\frac{m_3+tv_3}{|\mathbf{m}+t\mathbf{v}|}\right)^2\right]d\mathbf{x}' \\
&+\epsilon\int_{\Omega'}\left|\frac{\nabla\mathbf{m}+t\nabla\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}-\frac{(\mathbf{m}+t\mathbf{v})[(\mathbf{m}+t\mathbf{v})\cdot\nabla(\mathbf{m}+t\mathbf{v})]}{|\mathbf{m}+t\mathbf{v}|^3}\right|^2 d\mathbf{x}' \\
&+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,d\mathbf{x}' \\
&-\int_{\Omega'}\mathbf{h}_e\cdot\left(\frac{\mathbf{m}+t\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}\right)d\mathbf{x}'.
\end{aligned}$$

Then, we note that

$$\int_{\mathbb{R}^3}\nabla(\frac{\partial u}{\partial t})\cdot\nabla u\,dx=\int_{\Omega}\gamma'(t)\cdot\nabla u\,dx.$$

Consequently,

$$\begin{aligned}
I'[t] = &\int_{\Omega'}\gamma'(t)\cdot\nabla u\,d\mathbf{x}'-\int_{\Omega'}\mathbf{h}_e\cdot\gamma'(t)\,d\mathbf{x}' \\
&+2q\int_{\Omega'}\left(\frac{m_2+tv_2}{|\mathbf{m}+t\mathbf{v}|}\right)\left(\frac{v_2}{|\mathbf{m}+t\mathbf{v}|}-(m_2+tv_2)\frac{[(\mathbf{m}+t\mathbf{v})\cdot\mathbf{v}]}{|\mathbf{m}+t\mathbf{v}|^3}\right)d\mathbf{x}' \\
&+2q\int_{\Omega'}\left(\frac{m_3+tv_3}{|\mathbf{m}+t\mathbf{v}|}\right)\left(\frac{v_3}{|\mathbf{m}+t\mathbf{v}|}-(m_3+tv_3)\frac{[(\mathbf{m}+t\mathbf{v})\cdot\mathbf{v}]}{|\mathbf{m}+t\mathbf{v}|^3}\right)d\mathbf{x}' \\
&+2\epsilon\int_{\Omega'}\left[\frac{\nabla\mathbf{m}+t\nabla\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}-\frac{(\mathbf{m}+t\mathbf{v})[(\mathbf{m}+t\mathbf{v})\cdot\nabla(\mathbf{m}+t\mathbf{v})]}{|\mathbf{m}+t\mathbf{v}|^3}\right] \\
&\cdot\left\{\frac{\nabla\mathbf{v}}{|\mathbf{m}+t\mathbf{v}|}-\frac{[(\mathbf{m}+t\mathbf{v})\cdot\mathbf{v}](\nabla\mathbf{m}+t\nabla\mathbf{v})}{|\mathbf{m}+t\mathbf{v}|^3}\right. \\
&-\left(\frac{\mathbf{v}((\mathbf{m}+t\mathbf{v})\cdot\nabla(\mathbf{m}+t\mathbf{v}))}{|\mathbf{m}+t\mathbf{v}|^3}+\frac{(\mathbf{m}+t\mathbf{v})(\mathbf{v}\cdot\nabla(\mathbf{m}+t\mathbf{v})+(\mathbf{m}+t\mathbf{v})\cdot\nabla\mathbf{v})}{|\mathbf{m}+t\mathbf{v}|^3}\right. \\
&\left.\left.-3(\mathbf{m}+t\mathbf{v})((\mathbf{m}+t\mathbf{v})\cdot\nabla(\mathbf{m}+t\mathbf{v}))\cdot\frac{[(\mathbf{m}+t\mathbf{v})\cdot\mathbf{v}]}{|\mathbf{m}+t\mathbf{v}|^5}\right)\right\}d\mathbf{x}'.
\end{aligned}$$

Note, however, that

$$\gamma'(0)=\mathbf{v}-(\mathbf{m}\cdot\mathbf{v})\mathbf{m}.$$

Then,

$$\begin{aligned}
0=I'[0] &= \int_{\Omega'}[\mathbf{v}-(\mathbf{m}\cdot\mathbf{v})\mathbf{m}]\cdot\nabla u\,d\mathbf{x}'-\int_{\Omega'}\mathbf{h}_e\cdot[\mathbf{v}-(\mathbf{m}\cdot\mathbf{v})\mathbf{m}]\,d\mathbf{x}' \\
&+2q\int_{\Omega'}m_2\left(v_2-m_2(\mathbf{m}\cdot\mathbf{v})\right)d\mathbf{x}'+2q\int_{\Omega'}m_3\left(v_3-m_3(\mathbf{m}\cdot\mathbf{v})\right)d\mathbf{x}' \\
&+2\epsilon\int_{\Omega'}[\nabla\mathbf{m}-\mathbf{m}(\mathbf{m}\cdot\nabla\mathbf{m})]\cdot\left\{\nabla\mathbf{v}-(\mathbf{m}\cdot\mathbf{v})\nabla\mathbf{m}-\Big(\mathbf{v}(\mathbf{m}\cdot\nabla\mathbf{m})\right. \\
&\left.+\mathbf{m}(\mathbf{v}\cdot\nabla\mathbf{m}+\mathbf{m}\cdot\nabla\mathbf{v})-3\mathbf{m}(\mathbf{m}\cdot\nabla\mathbf{m})(\mathbf{m}\cdot\mathbf{v})\Big)\right\}d\mathbf{x}'.
\end{aligned}$$

Notice that

$$\int_{\Omega'}\nabla\mathbf{m}\cdot\nabla\mathbf{v}\,d\mathbf{x}'=\int_{\Omega'}\sum_{ij}\frac{\partial m_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}'=\sum_{ij}\int_{\Omega'}\frac{\partial m_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}'.$$

Integrating by part, we obtain

$$\int_{\Omega'} \nabla \mathbf{m} \cdot \nabla \mathbf{v} \, d\mathbf{x}' = -\int_{\Omega'} \sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} \cdot v_i \, d\mathbf{x}' = -\int_{\Omega'} \mathbf{\Delta m} \cdot \mathbf{v} \, d\mathbf{x}'$$

we also note that

$$\mathbf{m} \cdot \nabla \mathbf{m} = \sum_{ij} m_j \cdot \frac{\partial m_i}{\partial x_j},$$

and

$$\nabla \mathbf{m} \cdot \mathbf{m} = \sum_{ij} \frac{\partial m_i}{\partial x_j} \cdot m_i,$$

then,

$$\begin{aligned}
&2\epsilon \int_{\Omega'} [\nabla \mathbf{m} - \mathbf{m}(\mathbf{m} \cdot \nabla \mathbf{m})] \cdot \Big\{ \nabla \mathbf{v} - (\mathbf{m} \cdot \mathbf{v}) \nabla \mathbf{m} - \Big(\mathbf{v}(\mathbf{m} \cdot \nabla \mathbf{m}) \\
&+ \mathbf{m}(\mathbf{v} \cdot \nabla \mathbf{m} + \mathbf{m} \cdot \nabla \mathbf{v}) - 3\mathbf{m}(\mathbf{m} \cdot \nabla \mathbf{m})(\mathbf{m} \cdot \mathbf{v}) \Big) \Big\} \, d\mathbf{x}' \\
&= 2\epsilon \int_{\Omega'} -\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} v_i - (\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} v_i)(\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} m_i) - (\sum_{ij} \frac{\partial m_i}{\partial x_j} m_i)(\sum_{ij} \frac{\partial m_i}{\partial x_j} v_i) \\
&- (\sum_{ij} v_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} \frac{\partial m_i}{\partial x_j} m_i) - (\sum_{ij} m_j \frac{\partial v_i}{\partial x_j})(\sum_{ij} \frac{\partial m_i}{\partial x_j} m_i) \\
&+ 3(\sum_i m_i v_i)(\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} \frac{\partial m_i}{\partial x_j} m_i) - (\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} m_j \frac{\partial v_i}{\partial x_j}) \\
&- (\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} v_i)(\sum_{ij} m_j \frac{\partial m_i}{\partial x_j}) + (\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_i m_i v_i) \\
&+ (\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} v_j \frac{\partial m_i}{\partial x_j}) + (\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} m_j \frac{\partial v_i}{\partial x_j}) \\
&- 3(\sum_i m_i v_i)(\sum_{ij} m_j \frac{\partial m_i}{\partial x_j})(\sum_{ij} m_j \frac{\partial m_i}{\partial x_j}) \, d\mathbf{x}' \\
&= 2\epsilon \int_{\Omega'} -\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} v_i + (-\sum_{ij} \frac{\partial^2 m_i}{\partial x_j^2} m_i)(\sum_i m_i v_i) \, d\mathbf{x}' \\
&= 2\epsilon \int_{\Omega'} -\mathbf{\Delta m} \cdot \mathbf{v} + (\mathbf{\Delta m} \cdot \mathbf{m})(\mathbf{m} \cdot \mathbf{v}) \, d\mathbf{x}'.
\end{aligned}$$

Then,

$$\begin{aligned}
0 = I'[0] &= \int_{\Omega'} [\mathbf{v} - (\mathbf{m} \cdot \mathbf{v})\mathbf{m}] \cdot \nabla u \, d\mathbf{x}' - \int_{\Omega'} \mathbf{h}_e \cdot [\mathbf{v} - (\mathbf{m} \cdot \mathbf{v})\mathbf{m}] \, d\mathbf{x}' \\
&+ 2q \int_{\Omega'} m_2 \left(v_2 - m_2(\mathbf{m} \cdot \mathbf{v}) \right) \, d\mathbf{x}' + 2q \int_{\Omega'} m_3 \left(v_3 - m_3(\mathbf{m} \cdot \mathbf{v}) \right) \, d\mathbf{x}' \\
&+ 2\epsilon \int_{\Omega'} -\mathbf{\Delta m} \cdot \mathbf{v} + (\mathbf{\Delta m} \cdot \mathbf{m})(\mathbf{m} \cdot \mathbf{v}) \, d\mathbf{x}',
\end{aligned}$$

that is,

$$\int_{\Omega'} \tilde{\mathbf{h}_{\text{eff}}} \cdot \mathbf{v} = 0.$$

The above identity is valid for each function $\mathbf{v} \in C_c^\infty(\Omega')$. We then derive

$$\tilde{\mathbf{h}_{\text{eff}}} = \nabla u - (\mathbf{m} \cdot \nabla u)\mathbf{m} - \mathbf{h}_e + (\mathbf{h}_e \cdot \mathbf{m})\mathbf{m} + 2q[(m_2\mathbf{e}_2 + m_3\mathbf{e}_3) - (m_2^2 + m_3^2)\mathbf{m}] + 2\epsilon\Big[- \Delta\mathbf{m} + (\Delta\mathbf{m} \cdot \mathbf{m})\mathbf{m}\Big].$$

Note that

$$\mathbf{h}_{\text{eff}} = 2q(m_2\mathbf{e}_2 + m_3\mathbf{e}_3) - 2\epsilon\Delta\mathbf{m} + \nabla u - \mathbf{h}_e,$$

and

$$(\mathbf{h}_{\text{eff}} \cdot \mathbf{m})\mathbf{m} = 2q[(m_2\mathbf{e}_2 + m_3\mathbf{e}_3) \cdot \mathbf{m}]\mathbf{m} - 2\epsilon(\Delta\mathbf{m} \cdot \mathbf{m})\mathbf{m} + (\nabla u \cdot \mathbf{m})\mathbf{m} - (\mathbf{h}_e \cdot \mathbf{m})\mathbf{m},$$

we can compute for simplicity and then get

$$\tilde{\mathbf{h}_{\text{eff}}} = \mathbf{h}_{\text{eff}} - (\mathbf{h}_{\text{eff}}, \mathbf{m})\mathbf{m}.$$